

Limits and Derivatives

A derivative is a rate with units. Read it correctly and a 1% index move explains a 7.6% account move.

VOLUME I · Calculus for Finance

BUILDS ON LM1

LEADS TO LM3 · LM4 · LM5 · LM7 · LM31

TIME $\approx$ 2.5 h + lab

LEARNING OUTCOMES

Mastery *After this module you should be able to:*

☐	1 evaluate limits, including one-sided limits and limits at infinity, and identify where a function is discontinuous or not differentiable
☐	2 define the derivative as the limit of a difference quotient, and interpret it as a slope, a marginal rate and a sensitivity with units
☐	3 derive the power, exponential, logarithmic, product, quotient and chain rules, and apply them to financial functions
☐	4 calculate the delta of a futures position, the modified duration of a bond and the rate sensitivity of a present value
☐	5 calculate elasticities, and show that the elasticity of a futures account's equity with respect to the futures price equals its leverage, $\text{exposure} \div \text{equity}$
☐	6 estimate changes with differentials, bound the error of a linear approximation, and explain when it fails
☐	7 compute derivatives numerically with forward and central differences, and choose a step size that balances truncation and rounding error
☐	8 implement and verify derivatives in Python with finite differences and symbolic differentiation

LEARNING MODULE OVERVIEW

- A limit describes where a function is heading, not where it is. Two limits carry the whole module, $(e^x - 1)/x \rightarrow 1$ and $\ln(1 + x)/x \rightarrow 1$ as $x \rightarrow 0$, and both follow from one inequality and the squeeze theorem.
- The derivative $h'(x_0) = \lim_{\Delta x \rightarrow 0} [h(x_0 + \Delta x) - h(x_0)]/\Delta x$ is a rate with units: NT\$200 per index point for one TX contract, -0.0898 per 100 face per basis point for a 10-year 2% bond at par.
- Six rules — power, exponential, logarithm, product, quotient and chain — differentiate every function built from powers, exponentials and logarithms. Each is proved here from the definition.
- Modified duration is the derivative $-P'(y)/P$, and Macaulay duration falls out of it. For the 10-year 2% par bond $D_{\text{mod}} = 8.9826$, which equals its 10-year annuity factor at 2%.
- Elasticity $d \ln h / d \ln x$ is a unit-free derivative. A futures account's equity elasticity equals its leverage $f = \text{exposure} \div \text{equity}$: on 11 June 2025 one TX contract on NT\$600,000 had $f = 7.37$ and returned 7.60% on a 1.025% TAIEX day.
- The leverage of a fixed position is not constant: its own elasticity is $1 - f$. In April 2025 an account starting at $f = 4.73$ reached $f = 44.3$ in three sessions, while the linear approximation predicted 8.14.
- Forward differences have truncation error $O(\Delta x)$, central differences $O(\Delta x^2)$, and rounding adds an error of order $\varepsilon_m/\Delta x$. In double precision the best steps for the bond are near 10^{-8} and 10^{-6} ; a central 1 bp bump gets duration right to within 0.000002.

1

INTRODUCTION

A derivative answers one question: when an input moves a little, how much does an output move, per unit of input? Traders answer it daily without naming it. One TX contract gains NT\$200 per index point; a bond with modified duration 9 loses about 9% per percentage point of yield, to first order.

This module builds the mathematics behind such numbers — limits, derivatives and elasticities — and shows where the linear picture breaks.

MOTIVATING CASE A 1% day that becomes a 7.6% day

On 11 June 2025 TAIEX closed at 22,470.10, up 1.025%. The June TAIEX futures contract (TX) settled at 22,344, up 228 points. TX is marked to market at NT\$200 per index point, so each long contract was credited NT\$45,600 that evening. It was an ordinary session: 56 of the 145 sessions from 2 April to 31 October 2025 moved more.

Two traders were each long one contract at the previous day's settlement. Trader A had NT\$600,000 of account equity and earned +7.60% on the day. Trader B had NT\$900,000 and earned +5.07%. Both accounts exceeded an initial margin of NT\$350,000 per contract. That margin is an *illustrative assumption*: TAIFEX resets margins often, and Section 6 quotes a dated official figure.

Two months earlier the same position had been a disaster. On 7 April 2025, the largest daily fall of the sample, TX settled at its −10% daily limit. Trader A lost 71% of equity in one session and Trader B lost 47%.

Why is the multiple 7.4 for one trader and 4.9 for the other? Why does it not stay fixed? And why did a 19% fall in the futures price over three sessions lift B's leverage from 4.7 to 44?

Each answer is one derivative. Section 6 resolves the case, and Section 7 explains why the linear answer failed in April.

Exhibit 1: One TX contract, two accounts, two days

	11 June 2025	7 April 2025
TAIEX close: previous → current	22,242.14 → 22,470.10	21,298.22 → 19,232.35
TAIEX return	+1.025%	−9.700%
TX contract held	June 2025	April 2025
TX settlement: previous → current	22,116 → 22,344	21,296 → 19,167
TX return	+1.031%	−9.997%
Mark-to-market per contract (NT\$200 × points)	+NT\$45,600	−NT\$425,800
Notional at previous settlement (NT\$200 × F)	NT\$4,423,200	NT\$4,259,200
Account A (NT\$600,000): return on equity	+7.600%	−70.967%
Account B (NT\$900,000): return on equity	+5.067%	−47.311%
Account return ÷ TAIEX return: A, B	7.4, 4.9	7.3, 4.9

Source: FinMind `TaiwanFuturesDaily` (TX, regular session, daily settlement prices) and `TaiwanStockPrice` (TAIEX), retrieved 10 October 2026; TX multiplier from the TAIFEX contract specification. Equity is measured at the previous settlement. The market was closed on 3 and 4 April 2025, so the session before 7 April is 2 April.

Computations: `data/lm02_case.py` and the companion notebook `LM02_lab.ipynb`.

Sections 2–4 build the tools: limits, continuity, the derivative and its rules. Section 5 reads derivatives as sensitivities: futures delta, duration, rate risk. Section 6 introduces elasticity, which resolves the case.

Section 7 turns derivatives into approximations and shows how they fail. Section 8 computes derivatives numerically, where rounding makes "a smaller step is better" false, and Section 9 does it in Python.

NOTE Notation in this module

f is exposure ÷ equity, so $f > 1$ is leverage. Chosen to maximize growth $g(f)$, it is the Kelly fraction f^* of LM4; L is a leveraged fund's daily multiple (LM3). Generic functions are h and u of a variable x , as in $h(u(x))$; x_0 is a point, Δx a step and ℓ a limit.

Local symbols: N contracts with multiplier m (NT\$ per index point), futures price F , index S and equity E , so $f = NmF/E$; F_z is the futures price at which equity is zero. In $F = Se^{(r_f - q)\tau}$, r_f is the continuously compounded financing rate, q the continuous dividend yield and τ the time to expiry in years; T is a bond's or payment's maturity in years. $\eta_{h,x}$ is an elasticity, Q_+ , Q_- , Q_c are difference quotients and ε_m is machine epsilon. Counts: n periods or payments, j an integer power.

2 LIMITS AND CONTINUITY

- ☐ evaluate limits, including one-sided limits and limits at infinity, and identify where a function is discontinuous or not differentiable

Where a function is heading

The slope of a chord of e^x from 0 to x is $(e^x - 1)/x$. At $x = 0$ the expression is 0/0 and means nothing. For $x = 0.01$ it is 1.005017, and for $x = -0.01$ it is 0.995017: the values head toward 1 from both sides. A limit makes "heads toward" precise, and it does not care what happens *at* the point.

DEFINITION Limit

$\lim_{x \rightarrow x_0} h(x) = \ell$ means that $h(x)$ is as close to ℓ as we like whenever x is close enough to x_0 , with $x \neq x_0$. Precisely: for every $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|h(x) - \ell| < \varepsilon$. The **one-sided limits** $\lim_{x \rightarrow x_0^-}$ and $\lim_{x \rightarrow x_0^+}$ let x approach from one side only; the limit exists exactly when both exist and agree. $\lim_{x \rightarrow \infty} h(x) = \ell$ means that for every $\varepsilon > 0$ there is a number x_ε such that $x > x_\varepsilon$ implies $|h(x) - \ell| < \varepsilon$.

Limits combine as expected: if $\lim h = \ell_1$ and $\lim u = \ell_2$, then $\lim(h \pm u) = \ell_1 \pm \ell_2$, $\lim hu = \ell_1 \ell_2$, and $\lim h/u = \ell_1/\ell_2$ when $\ell_2 \neq 0$. These **limit laws** are stated without proof (Stewart, chapter 2). One more tool does the heavy lifting below.

THEOREM Squeeze theorem

If $h_1(x) \leq h(x) \leq h_2(x)$ for all $x \neq x_0$ near x_0 , and $\lim_{x \rightarrow x_0} h_1(x) = \lim_{x \rightarrow x_0} h_2(x) = \ell$, then $\lim_{x \rightarrow x_0} h(x) = \ell$.

DEEP DIVE The ε - δ definition at work optional · click to expand

A limit from the definition. To show $\lim_{x \rightarrow 2} (3x - 1) = 5$, let $\varepsilon > 0$ and choose $\delta = \varepsilon/3$. Then $0 < |x - 2| < \delta$ gives $|(3x - 1) - 5| = 3|x - 2| < 3\delta = \varepsilon$. The proof is a recipe: given a tolerance ε on the output, produce a tolerance δ on the input.

Proof of the squeeze theorem. Let $\varepsilon > 0$. Choose δ so small that $0 < |x - x_0| < \delta$ implies $h_1(x) > \ell - \varepsilon$ and $h_2(x) < \ell + \varepsilon$; both are possible because h_1 and h_2 tend to ℓ . On that set $\ell - \varepsilon < h_1(x) \leq h(x) \leq h_2(x) < \ell + \varepsilon$, so $|h(x) - \ell| < \varepsilon$. ■

Continuity

DEFINITION Continuity

h is **continuous at** x_0 if $h(x_0)$ is defined, $\lim_{x \rightarrow x_0} h(x)$ exists, and the two are equal. h is continuous on an interval if it is continuous at each of its points.

Polynomials, e^x and $\ln x$ (for $x > 0$) are continuous. So are sums, products, quotients and compositions of continuous functions, wherever they are defined. Continuity also lets a limit pass inside a function: if $\lim_{x \rightarrow x_0} u(x) = \ell$ and h is continuous at ℓ , then $\lim_{x \rightarrow x_0} h(u(x)) = h(\ell)$. All three facts are stated without proof (Stewart, chapter 2).

A function can fail to be continuous in three ways, and each has a financial example.

- **Removable discontinuity.** $(e^x - 1)/x$ is undefined at 0, but its limit exists (it is 1, as shown below). Defining the value at 0 as 1 repairs it.
- **Jump.** A digital option pays 1 if the index ends above the strike K and 0 otherwise. Its payoff jumps at K , where the left and right limits are 0 and 1.
- **Infinite discontinuity.** The leverage $f = NmF/E$ of a futures account grows without bound as equity approaches zero. For Trader A on 11 June, equity is zero at $F = 22,116 - 600,000/200 = 19,116$, and $f \rightarrow +\infty$ as F falls toward that level.

Continuity is weaker than smoothness. The call payoff $\max(S - K, 0)$ is continuous at the strike but has a corner there; Section 3 shows that it has no derivative at K .

Two limits that carry the module

By (1.6) of LM1, $e^x = \lim_{n \rightarrow \infty} (1 + x/n)^n$ for every real x . Everything we need about the slope of e^x then follows from **Bernoulli's inequality**: for every integer $n \geq 1$ and every $x \geq -1$,

$$(1+x)^n \geq 1+nx.$$

The proof is by induction. It holds with equality for $n = 1$. If it holds for n , then, because $1+x \geq 0$, $(1+x)^{n+1} \geq (1+nx)(1+x) = 1+(n+1)x+nx^2 \geq 1+(n+1)x$.

Replace x by x/n with $n \geq |x|$: $(1+x/n)^n \geq 1+x$. Letting $n \rightarrow \infty$ preserves the inequality, so $e^x \geq 1+x$ for every real x . Applied to $-x$ it gives $e^{-x} \geq 1-x$, which is positive when $x < 1$. Since $e^x e^{-x} = 1$ by the laws of exponents (LM1),

$$1+x \leq e^x \leq \frac{1}{1-x} \quad (x < 1). \quad (2.1)$$

Subtract 1 and divide by x . For $0 < x < 1$ this gives $1 \leq (e^x - 1)/x \leq 1/(1-x)$. For $x < 0$ the division flips both inequalities: $1/(1-x) \leq (e^x - 1)/x \leq 1$. In both cases the bounds tend to 1 (Exhibit 2 shows them closing in), so the squeeze theorem yields

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1. \quad (2.2)$$

A second limit follows by substitution. Let $z = \ln(1+x)$, so that $x = e^z - 1$. As $x \rightarrow 0$, $z \rightarrow 0$ because $\ln$ is continuous at 1, and $z \neq 0$ whenever $x \neq 0$. Then $\ln(1+x)/x = z/(e^z - 1) = 1/[(e^z - 1)/z]$, and the quotient law with (2.2) gives

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1. \quad (2.3)$$

Exhibit 2: Squeezing the slope of e^x at zero

x	$(e^x - 1)/x$	Lower bound	Upper bound	$\ln(1+x)/x$
+0.100	1.051709	1	1.111111	0.953102
+0.010	1.005017	1	1.010101	0.995033
+0.001	1.000500	1	1.001001	0.999500
-0.001	0.999500	0.999001	1	1.000500
-0.010	0.995017	0.990099	1	1.005034
-0.100	0.951626	0.909091	1	1.053605

Note: the bounds are 1 and $1/(1-x)$ from (2.1). For $x > 0$ the first ratio approaches 1 from above and the second from below; for $x < 0$ the sides swap.

In financial language, (2.2) says that a continuously compounded rate x produces a simple return $e^x - 1$ whose ratio to x tends to 1. Equation (2.3) says the same of a log return and a simple return. Both are " $\approx$ " statements about small moves; LM3 measures their error.

Limits at infinity

Limits at infinity describe long horizons, such as the value of an income stream that never stops.

EXAMPLE 1

From annuity to perpetuity

An annuity pays 1 at the end of each of n years. At the periodic rate $i = 2\%$ its present value is the annuity factor $a_{n,i} = (1 - (1 + i)^{-n})/i$, which LM6 derives. Compute $a_{n,i}$ for $n = 10, 30, 100$ and 1,000, and find its limit as $n \rightarrow \infty$.

SOLUTION

The values are 8.9826, 22.3965, 43.0984 and 50.0000. Because $0 < 1/1.02 < 1$, the term 1.02^{-n} tends to 0, and the limit laws give $\lim_{n \rightarrow \infty} a_{n,i} = 1/0.02 = 50$: the value of a **perpetuity**.

The gap $50 - a_{n,i} = 50 \times 1.02^{-n}$ shrinks by the same factor $1/1.02$ every year, so convergence is steady but slow. After 100 years, $50 - 43.0984 = 6.9016$, or 13.80% of the perpetuity's value, is still missing. The 10-year factor, 8.9826, will reappear in Section 5 as the duration of a 10-year bond.

PITFALL Market data have no limits

Prices move in ticks (one TX tick is one index point, NT\$200), print at discrete times, and gap. On 7 April 2025 TX settled at its -10% daily limit, 2,129 points below a settlement struck five calendar days earlier. Limits and derivatives belong to a *model* — a pricing formula, a fitted curve, an account-value function — not to a raw price series. "The derivative of the price path" is a modelling choice and should be stated as one.

導讀 極限：看「趨近」，不看「等於」

極限描述的是函數「往哪裡去」，不在乎那一點本身有沒有定義。 $(e^x - 1)/x$ 在 $x = 0$ 是 $0/0$ ，但從兩邊都被夾到 1（夾擠定理 squeeze theorem），這就是「連續複利 x 很小時，簡單報酬 $e^x - 1 \approx x$ 」的嚴格版本。不連續有三種：可移除（ $(e^x - 1)/x$ 在 0）、跳躍（二元選擇權在履約價）、無窮（權益趨近 0 時的槓桿）；買權在履約價連續但不可微。記住：市場資料是一格一格跳的，極限與導數是「模型」的性質，不是報價本身的性質。

KNOWLEDGE CHECK 1

Use (2.2) to evaluate $\lim_{\Delta t \rightarrow 0} (e^{0.05\Delta t} - 1)/\Delta t$. What does the answer say about a 5% continuously compounded rate over a short period of Δt years?

ANSWER

Put $z = 0.05\Delta t$: the expression equals $0.05(e^z - 1)/z$, which tends to $0.05 \cdot 1 = 0.05$. Over a short period of Δt years the simple return $e^{0.05\Delta t} - 1$ is $0.05\Delta t$ up to an error of order Δt^2 , between 0 and $z^2/(1 - z)$ by (2.1). The instantaneous growth rate of the investment is 5% a year.

3 THE DERIVATIVE

- ☐ define the derivative as the limit of a difference quotient, and interpret it as a slope, a marginal rate and a sensitivity with units

From average to instantaneous rate

The average rate of change of h between x_0 and $x_0 + \Delta x$ is the **difference quotient** $[h(x_0 + \Delta x) - h(x_0)]/\Delta x$: the slope of the secant through the two points. As Δx shrinks, the secant pivots about $(x_0, h(x_0))$. If it settles on a limiting line, that line is the tangent and its slope is the derivative.

DEFINITION Derivative

h is **differentiable** at x_0 if the finite limit

$$h'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x} \quad (2.4)$$

exists. $h'(x_0)$ is the **derivative** of h at x_0 , also written $\frac{dh}{dx}|_{x=x_0}$. If h is differentiable at every point of an interval, $x \mapsto h'(x)$ is a new function; differentiating it again gives the second derivative h'' .

A derivative carries units: output units per input unit. That is why one number answers three questions.

- **Slope** (geometry): the steepness of the tangent line.
- **Marginal rate** (economics): the extra output from one more unit of input, such as the extra interest from lengthening a loan.
- **Sensitivity** (risk): the change in a value per unit change in a risk factor. A TX contract has a sensitivity of NT\$200 per index point.

Two derivatives from the definition show the mechanics. For $h(x) = x^2$, $[(x + \Delta x)^2 - x^2]/\Delta x = 2x + \Delta x \rightarrow 2x$. For $h(x) = 1/x$ with $x \neq 0$, $[1/(x + \Delta x) - 1/x]/\Delta x = -1/[x(x + \Delta x)] \rightarrow -1/x^2$.

EXAMPLE 2

A derivative from the definition: the one-year zero

A one-year zero-coupon bond pays 100 at maturity, so its price at annual yield y is $P(y) = 100/(1 + y)$. Find $P'(y)$ from (2.4) and evaluate it at $y = 2\%$. How much does the price fall per basis point for NT\$100 million of face value?

SOLUTION

The difference quotient is

$$\frac{1}{\Delta y} \left[\frac{100}{1 + y + \Delta y} - \frac{100}{1 + y} \right] = \frac{100 [(1 + y) - (1 + y + \Delta y)]}{\Delta y (1 + y)(1 + y + \Delta y)} = \frac{-100}{(1 + y)(1 + y + \Delta y)}.$$

As $\Delta y \rightarrow 0$ it tends to $P'(y) = -100/(1 + y)^2$. At $y = 2\%$, $P'(0.02) = -96.1169$ per 100 face per unit of yield.

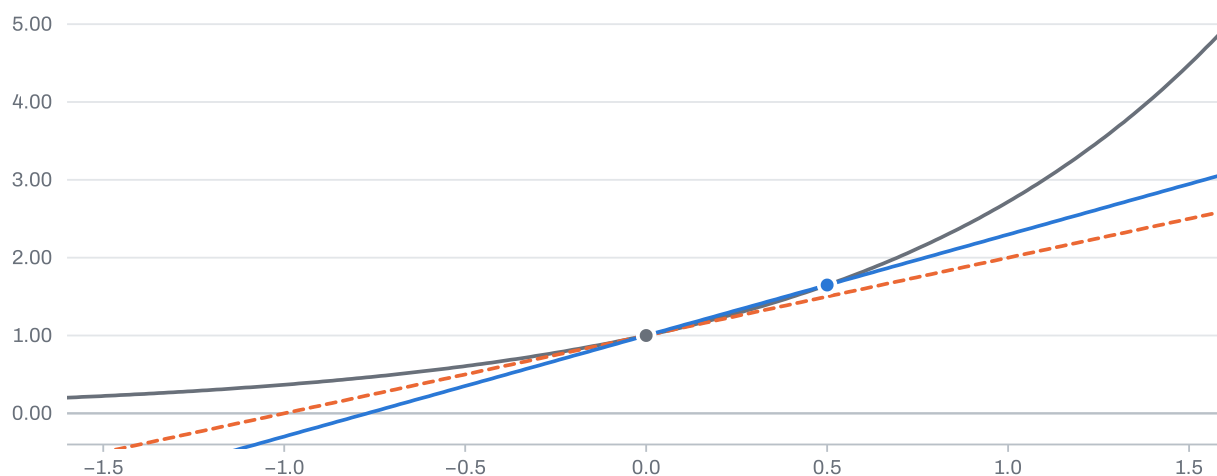
A basis point is 0.0001 units of yield, so the price falls by 0.009612 per 100 face per bp. For NT\$100 million of face, that is NT\$9,612 per bp. This number — the price fall for a one-basis-point rise in yield — is called the **DV01**.

The figure below lets you watch the secant become the tangent. Its opening view uses $h(x) = e^x$ at $x_0 = 0$ with a forward step $\Delta x = 0.5$: the secant slope is 1.29744 while the tangent slope is 1. Shrink Δx and the error falls in proportion to Δx ; switch to central differences and it falls in proportion to Δx^2 .

From secant to tangent

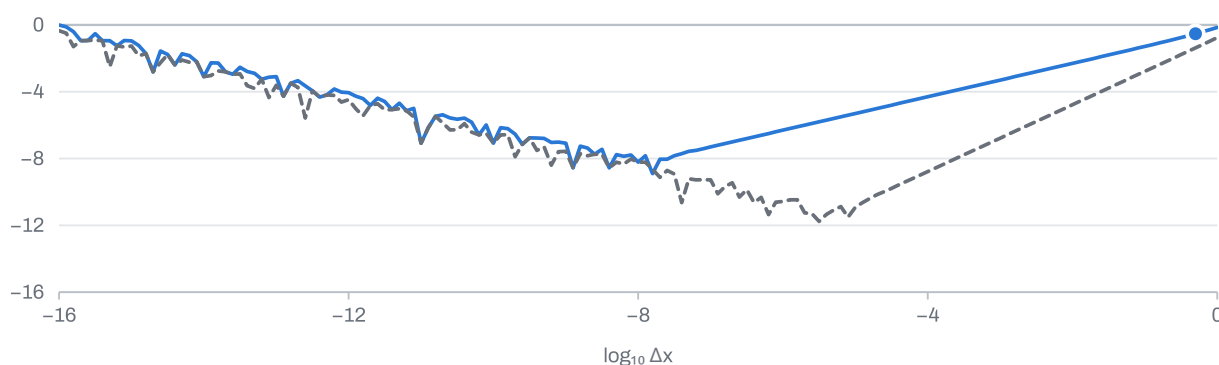
INTERACTIVE

— e^x - - - Tangent at x_0 — Secant



— Forward (selected) - - - Central

$\log_{10}$ relative error



Difference quotient

1.29744

Derivative $h'(x_0)$

1.00000

Error

0.29744

Error $\div \Delta x$

0.59489

$x_0 = 0$, so the exact derivative is $e^0 = 1$.

Upper panel — grey: the function; dashed: the tangent at x_0 , whose slope is the derivative (when one exists); solid: the secant, whose slope is the difference quotient. Lower panel — error of the quotient against the step Δx on log-log axes, computed live in double precision: slope 1 means first order, slope 2 second order, and the rise on the left is rounding error.

Real pricing functions behave the same way. Exhibit 3 applies (2.4) to the price of a 10-year bond with a 2% annual coupon, valued at a 2% yield, where it trades at par. Section 5 derives the exact derivative, $P'(2\%) = -898.2585$.

Exhibit 3: Difference quotients of a bond price at $y_0 = 2\%$

Step Δy	Forward quotient	Error	Central quotient	Error
100 bp	-853.0203	+45.2382	-900.0754	-1.816868
10 bp	-893.5768	+4.6817	-898.2767	-0.018153
1 bp	-897.7887	+0.4698	-898.2587	-0.000182
0.1 bp	-898.2115	+0.0470	-898.2585	-0.000002
0.01 bp	-898.2538	+0.0047	-898.2585	0.000000

Note: 10-year bond, 2% annual coupon, price per 100 face; the quotients are price change per unit of yield. Forward: $[P(y_0 + \Delta y) - P(y_0)]/\Delta y$. Central: $[P(y_0 + \Delta y) - P(y_0 - \Delta y)]/(2\Delta y)$. Each tenfold cut in Δy cuts the forward error about tenfold and, until rounding takes over at 0.01 bp, the central error about a hundredfold; Section 8 explains why.

Differentiable implies continuous

If h is differentiable at x_0 , then $h(x_0 + \Delta x) - h(x_0) = \Delta x \cdot \frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x} \rightarrow 0 \cdot h'(x_0) = 0$, so h is continuous at x_0 . The converse fails.

The **one-sided derivatives** $h'_\pm(x_0) = \lim_{\Delta x \rightarrow 0^\pm} [h(x_0 + \Delta x) - h(x_0)]/\Delta x$ can exist and differ. For the call payoff $\max(S - K, 0)$ the right derivative at K is 1 and the left derivative is 0. The payoff therefore has no derivative at K . The digital payoff is not even continuous at K , so it cannot be differentiable there.

The mean value theorem

THEOREM Mean value theorem

If h is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) , there is a point $\xi \in (x_1, x_2)$ with

$$h'(\xi) = \frac{h(x_2) - h(x_1)}{x_2 - x_1}.$$

Somewhere, the instantaneous rate equals the average rate. When the bond's yield rises from 2% to 3%, its price falls from 100 to 91.469797, an average slope of -853.0203 per unit of yield. The theorem guarantees a yield between 2% and 3% at which P' takes exactly that value.

Two consequences are used repeatedly. If $h' > 0$ on an interval, h is increasing there. If $|h'| \leq M$ on $[x_1, x_2]$, then $|h(x_2) - h(x_1)| \leq M |x_2 - x_1|$, which is how Sections 7 and 8 bound errors.

DEEP DIVE Proof of the mean value theorem optional · click to expand

Rolle's theorem. Suppose also that $h(x_1) = h(x_2)$. A continuous function on a closed interval attains a maximum and a minimum (the extreme value theorem, stated without proof).

If both occur at the endpoints, h is constant and $h' = 0$ everywhere. Otherwise an extremum, say a maximum, occurs at an interior point ξ . There $[h(\xi + \Delta x) - h(\xi)]/\Delta x \leq 0$ for $\Delta x > 0$ and ≥ 0 for $\Delta x < 0$, so the limit $h'(\xi)$ is both ≤ 0 and ≥ 0 : $h'(\xi) = 0$.

The general case. Apply Rolle's theorem to $H(x) = h(x) - h(x_1) - \frac{h(x_2) - h(x_1)}{x_2 - x_1}(x - x_1)$, which satisfies $H(x_1) = H(x_2) = 0$. Some $\xi \in (x_1, x_2)$ has $H'(\xi) = h'(\xi) - \frac{h(x_2) - h(x_1)}{x_2 - x_1} = 0$. ■

PITFALL Per unit, per percent, per basis point

$P'(2\%) = -898.2585$ is a price change per *unit* of yield, that is, per 100 percentage points. Per percentage point it is -8.98 ; per basis point it is -0.0898 . Futures delta has the same trap: NT\$200 per index point for one TX contract, but NT\$44,688 per 1% move at $F = 22,344$. Most sensitivity errors in practice are unit errors, so write the units next to every derivative.

KNOWLEDGE CHECK 2

Use the definition (2.4) to find the derivative of $h(x) = x^3$ at $x = 2$.

ANSWER

$(2 + \Delta x)^3 - 8 = 12\Delta x + 6\Delta x^2 + \Delta x^3$, so the difference quotient is $12 + 6\Delta x + \Delta x^2$, which tends to 12. Hence $h'(2) = 12$, in agreement with the power rule of Section 4.

4 DIFFERENTIATION RULES

- ☐ derive the power, exponential, logarithmic, product, quotient and chain rules, and apply them to financial functions

Six rules, each proved once from (2.4), differentiate every function built from powers, exponentials and logarithms by sums, products, quotients and compositions.

Linearity and integer powers

From the limit laws, $(\kappa h)' = \kappa h'$ for any constant κ , and $(h + u)' = h' + u'$: differentiation is **linear**. For a positive integer j , the binomial theorem gives $(x + \Delta x)^j = x^j + j x^{j-1} \Delta x + \Delta x^2 \times$ (a polynomial in x and Δx). Dividing by Δx and letting $\Delta x \rightarrow 0$ leaves $\frac{d}{dx} x^j = j x^{j-1}$.

The exponential and the logarithm

PROPOSITION Derivatives of $e^{\kappa x}$ and $\ln x$

For any constant κ ,

$$\frac{d}{dx} e^{\kappa x} = \kappa e^{\kappa x} \quad \text{for all } x, \quad \frac{d}{dx} \ln x = \frac{1}{x} \quad \text{for } x > 0.$$

Proof. For $\kappa \neq 0$, the law of exponents gives $[e^{\kappa(x+\Delta x)} - e^{\kappa x}]/\Delta x = \kappa e^{\kappa x} (e^{\kappa \Delta x} - 1)/(\kappa \Delta x)$, which tends to $\kappa e^{\kappa x}$ by (2.2) with $\kappa \Delta x$ in place of x . For $\kappa = 0$ both sides are 0.

For the logarithm, $\ln(x + \Delta x) - \ln x = \ln(1 + \Delta x/x)$. With $z = \Delta x/x$,

$$\frac{\ln(x + \Delta x) - \ln x}{\Delta x} = \frac{1}{x} \cdot \frac{\ln(1 + z)}{z} \longrightarrow \frac{1}{x} \cdot 1$$

by (2.3), since $z \rightarrow 0$ as $\Delta x \rightarrow 0$. ■

With $\kappa = 1$, the exponential is its own rate of change. Wealth $W(t) = W_0 e^{gt}$ grows at $W'(t) = g W(t)$: a constant *proportional* rate g , the log growth rate of LM1. Its logarithm, $\ln W(t) = \ln W_0 + g t$, is a straight line with slope g .

Products and quotients

PROPOSITION Product and quotient rules

If u and h are differentiable at x , then $(uh)' = u'h + uh'$, and, where $h(x) \neq 0$, $(u/h)' = (u'h - uh')/h^2$.

Proof. Add and subtract $u(x)h(x + \Delta x)$:

$$\frac{u(x + \Delta x)h(x + \Delta x) - u(x)h(x)}{\Delta x} = \frac{u(x + \Delta x) - u(x)}{\Delta x} h(x + \Delta x) + u(x) \frac{h(x + \Delta x) - h(x)}{\Delta x}.$$

Because h is differentiable, it is continuous, so $h(x + \Delta x) \rightarrow h(x)$ and the right side tends to $u'h + uh'$. For the quotient, first

$$\frac{1}{\Delta x} \left[\frac{1}{h(x + \Delta x)} - \frac{1}{h(x)} \right] = -\frac{h(x + \Delta x) - h(x)}{\Delta x} \cdot \frac{1}{h(x + \Delta x)h(x)} \longrightarrow -\frac{h'}{h^2},$$

and then the product rule applied to $u \cdot (1/h)$ gives $u'/h - uh'/h^2 = (u'h - uh')/h^2$. ■

The chain rule

Most pricing functions are nested: a bond price depends on discount factors, which depend on the yield. The chain rule differentiates compositions.

THEOREM Chain rule

If u is differentiable at x and h is differentiable at $u(x)$, then $h(u(x))$ is differentiable at x and

$$\frac{d}{dx}h(u(x)) = h'(u(x)) u'(x), \quad \text{or} \quad \frac{dh}{dx} = \frac{dh}{du} \cdot \frac{du}{dx}. \quad (2.5)$$

The intuitive argument writes $\Delta h/\Delta x = (\Delta h/\Delta u)(\Delta u/\Delta x)$ and lets $\Delta x \rightarrow 0$. It is correct whenever $\Delta u \neq 0$ for small Δx , which can fail if u oscillates. The Deep Dive gives a proof without that gap.

DEEP DIVE A proof of the chain rule without dividing by zero optional · click to expand

Lemma (Carathéodory). h is differentiable at x_0 if and only if $h(x) - h(x_0) = \bar{h}(x)(x - x_0)$ near x_0 for some function $\bar{h}$ continuous at x_0 . Then $\bar{h}(x_0) = h'(x_0)$. If h is differentiable, set $\bar{h}(x) = [h(x) - h(x_0)]/(x - x_0)$ for $x \neq x_0$ and $\bar{h}(x_0) = h'(x_0)$; continuity of $\bar{h}$ at x_0 is the definition (2.4). Conversely, dividing by $x - x_0$ and letting $x \rightarrow x_0$ gives $h'(x_0) = \bar{h}(x_0)$.

Proof of (2.5). Let $z_0 = u(x_0)$. Write $u(x) - u(x_0) = \bar{u}(x)(x - x_0)$ and $h(z) - h(z_0) = \bar{h}(z)(z - z_0)$, with $\bar{u}$ continuous at x_0 and $\bar{h}$ continuous at z_0 . Then

$$h(u(x)) - h(u(x_0)) = \bar{h}(u(x))(u(x) - u(x_0)) = \bar{h}(u(x))\bar{u}(x)(x - x_0).$$

The factor $\bar{h}(u(x))\bar{u}(x)$ is continuous at x_0 , because u is continuous at x_0 , and its value there is $\bar{h}(z_0)\bar{u}(x_0) = h'(u(x_0))u'(x_0)$. By the lemma, $h(u(x))$ is differentiable at x_0 with that derivative. ■

The chain rule completes the toolkit.

- **Real powers.** For $x > 0$, $x^\alpha = e^{\alpha \ln x}$, so $\frac{d}{dx}x^\alpha = e^{\alpha \ln x} \cdot \alpha/x = \alpha x^{\alpha-1}$ for any real α .
- **Other bases.** For a base $\beta > 0$, $\beta^x = e^{x \ln \beta}$, so $\frac{d}{dx}\beta^x = \beta^x \ln \beta$.
- **Discount factors.** $\frac{d}{dy}(1+y)^{-t} = -t(1+y)^{-t-1}$ and $\frac{d}{dr}e^{-rt} = -te^{-rt}$.
- **Logarithmic derivative.** For $h > 0$, $\frac{d}{dx} \ln h(x) = h'(x)/h(x)$: the *relative* rate of change, the basis of elasticities in Section 6.

Exhibit 4 collects the rules with the places they appear in this book.

Exhibit 4: Differentiation rules and where they appear

Function	Derivative	Where it appears
x^α	$\alpha x^{\alpha-1}$	discount factors $(1+y)^{-t}$, annualizing
$e^{\kappa x}$	$\kappa e^{\kappa x}$	continuous compounding and discounting
β^x	$\beta^x \ln \beta$	discrete compounding as a function of time
$\ln x$	$1/x$	log returns, growth rates, Kelly growth
$u h$	$u' h + u h'$	forward rates, notional = NmF
u/h	$(u' h - u h')/h^2$	leverage NmF/E , current yield
$h(u(x))$	$h'(u(x)) u'(x)$	bond prices, currency translation
$\ln h(x)$	$h'(x)/h(x)$	relative rates, elasticities

EXAMPLE 3

Forward rates from a yield curve

A continuously compounded zero-coupon yield curve is $y(T) = 2.0\% - 0.8\% e^{-T/3}$ for maturity T in years (an illustrative curve, not market data). The discount factor is $P(T) = e^{-y(T)T}$. The instantaneous forward rate $r(T) := -\frac{d}{dT} \ln P(T)$ is the marginal rate earned by extending a deposit's maturity beyond T . Find $r(T)$ and evaluate it at $T = 1, 5$ and 10 .

SOLUTION

Since $\ln P(T) = -y(T)T$, the product rule gives $r(T) = y(T) + T y'(T)$. The derivative of $e^{-T/3}$ is $-\frac{1}{3}e^{-T/3}$, so $y'(T) = \frac{0.8\%}{3}e^{-T/3}$.

T	$y(T)$	$y'(T)$ per year	$r(T) = y + T y'$
1	1.42677%	0.191075%	1.61785%
5	1.84890%	0.050367%	2.10073%
10	1.97146%	0.009513%	2.06659%

The forward rate exceeds the yield wherever the curve slopes up. The yield is an *average* rate over $[0, T]$; the forward rate is the *marginal* rate at T . Whenever a total equals T times an average, $\text{marginal} = \text{average} + T \times \text{slope of the average}$.

EXAMPLE 4

How leverage changes with the futures price

An account holds N contracts with multiplier m , opened at F_0 with equity E_0 . Its equity is $E(F) = E_0 + Nm(F - F_0)$ and its leverage, exposure $\div$ equity, is $f(F) = NmF/E(F)$. Use the quotient rule to find $f'(F)$. Evaluate it for Trader A on 10 June 2025 ($N = 1$, $m = 200$, $F_0 = 22,116$, $E_0 = \text{NT\$}600,000$).

SOLUTION

The quotient rule gives

$$f'(F) = \frac{NmE - NmF \cdot Nm}{E^2} = \frac{Nm(E - NmF)}{E^2} = \frac{f}{F}(1 - f).$$

For Trader A, $f = 4,423,200/600,000 = 7.3720$, so $f'(F_0) = -7.3720 \times 6.3720/22,116 = -0.0021240$ per index point. A 100-point rise should cut leverage by about 0.2124. Exact repricing gives $f(22,216) = 4,443,200/620,000 = 7.1665$, a fall of 0.2055.

For $f > 1$ the derivative is negative: a fixed long position deleverages as the market rises and leverages up as it falls. Section 6 turns this into a one-line elasticity, and Section 7 shows how it misbehaves in a crash.

導讀 連鎖律：敏感度相乘

很多定價函數是「一層套一層」：債券價格 $\rightarrow$ 折現因子 $\rightarrow$ 殖利率；台幣價值 $\rightarrow$ 美元價值 $\times$ 匯率。連鎖律說：外層對內層的敏感度，乘上內層對變數的敏感度，就是整體的敏感度。乘法法則與除法法則看起來是代數，其實是「邊際 = 平均 + 期限 $\times$ 平均的斜率」（遠期利率）和「槓桿隨價格變動」這類金融結論的來源。後面的 Greeks、存續期間與 LM28 的 Itô 引理，全部建立在連鎖律之上。

KNOWLEDGE CHECK 3

Differentiate $\ln(1 + fR)$ with respect to f , treating the return R as a constant. What is the derivative at $f = 0$?

ANSWER

By the chain rule, $\frac{d}{df} \ln(1 + fR) = R/(1 + fR)$, which equals R at $f = 0$. Averaged over the possible returns, this derivative is $g'(f)$, the slope of the growth curve $g(f)$ of LM4; its zero is the Kelly fraction f^* .

- ☐ calculate the delta of a futures position, the modified duration of a bond and the rate sensitivity of a present value

In risk management a derivative is a **sensitivity**: the change in a value per unit change in one risk factor, other factors held fixed. This section computes three that desks quote daily: futures delta, bond duration and the rate sensitivity of a present value.

Delta of a futures position

An account holding N contracts opened at F_0 with equity E_0 has equity $E(F) = E_0 + Nm(F - F_0)$ after marking to market. Its derivative is the position's **delta**:

$$E'(F) = Nm \quad \text{NT\$ per index point.}$$

The sensitivity is constant because E is linear in F . By linearity of the derivative, the delta of several positions is the sum of their deltas. Exhibit 5 lists the three TAIEX futures contracts.

Exhibit 5: TAIEX futures contracts: multiplier and delta

Contract	Code	Multiplier (NT\$ per index point)	Tick	Delta of one long contract
TAIEX Futures	TX	200	1 point = NT\$200	NT\$200 per point
Mini-TAIEX Futures	MTX	50	1 point = NT\$50	NT\$50 per point
Micro TAIEX Futures	TMF	10	1 point = NT\$10	NT\$10 per point

Source: TAIEX contract specifications for TX, MTX and TMF (taifex.com.tw, retrieved 10 October 2026).

Delta per index point converts to delta per 1% move by multiplying by $F/100$. At $F = 22,344$ a 1% move is 223.44 points, worth NT\$44,688 to one TX contract. The quantity NmF is the position's **notional**, or exposure, and "NT\$ per 1%" is notional $\div 100$.

EXAMPLE 5

Delta of a mixed position

A trader is long 2 TX, short 3 MTX and long 4 TMF, all in the same expiry month. Compute the position's delta, its profit on a 150-point rise, its delta per 1% move at $F = 22,344$, and the TMF trade that makes it delta-neutral.

SOLUTION

Delta is $2 \times 200 - 3 \times 50 + 4 \times 10 = 290$ NT\$ per point, the same exposure as 1.45 TX contracts. A 150-point rise earns $150 \times 290 = \text{NT\$}43,500$. Per 1% move, delta is $290 \times 223.44 = \text{NT\$}64,798$. Selling 29 TMF ($29 \times 10 = 290$) brings delta to zero. All of this is exact, because each contract's value is linear in F .

Duration: the derivative of a bond price

A bond with annual cash flows CF_t at $t = 1, \dots, n$ is priced at yield y as $P(y) = \sum_t CF_t (1+y)^{-t}$. By linearity and the discount-factor rule,

$$P'(y) = - \sum_{t=1}^n t CF_t (1+y)^{-t-1} = - \frac{1}{1+y} \sum_{t=1}^n t PV_t, \quad PV_t := CF_t (1+y)^{-t}.$$

Dividing by $-P$ defines the **modified duration** and reveals the **Macauley duration** inside it:

$$D_{\text{mod}} := - \frac{P'(y)}{P(y)} = \frac{D_{\text{Mac}}}{1+y}, \quad D_{\text{Mac}} := \sum_{t=1}^n t w_t, \quad w_t := \frac{PV_t}{P}. \quad (2.6)$$

Macauley duration is the present-value-weighted average time of the cash flows. It is not an extra definition: it is what the derivative of the price function produces.

As a sensitivity, (2.6) says $\Delta P/P \approx -D_{\text{mod}} \Delta y$, with an error of order Δy^2 (Section 7). The price fall for a one-basis-point rise in yield, $-P'(y) \times 0.0001$, is the bond's **DV01**. LM3 adds the second-order term, convexity.

EXAMPLE 6

Duration of the 10-year bond as a derivative

The 10-year bond of Exhibit 3 pays a 2% annual coupon and yields 2%. Compute $P'(2\%)$, D_{mod} , D_{Mac} and the DV01 for NT\$100 million of face. Then compare the first-order estimate of the price change with exact repricing for yield changes of ± 25 bp.

SOLUTION

With coupons of 2 and principal 100, $P(0.02) = 100$ and $P'(0.02) = -\sum_{t=1}^{10} 2t/1.02^{t+1} - 1000/1.02^{11} = -898.2585$. So $D_{\text{mod}} = 8.982585$ and $D_{\text{Mac}} = 1.02 \times 8.982585 = 9.1622$ years; the final payment alone carries weight $w_{10} = 0.8368$. The DV01 is 0.08983 per 100 face, or NT\$89,826 per NT\$100 million.

For $\Delta y = +25$ bp the first-order estimate is $-8.982585 \times 0.0025 = -2.2456\%$; exact repricing gives -2.2166% . For -25 bp the estimate is $+2.2456\%$ and the exact change $+2.2753\%$. The exact changes are asymmetric, while the linear estimate is symmetric and too pessimistic on both sides, by about 3 bp of price. That asymmetry is convexity, the subject of LM3.

The duration 8.9826 equals the 10-year annuity factor of Example 1. This is no coincidence: a bond at par has $D_{\text{mod}} = (1 - (1 + y)^{-n})/y$ (Practice Problem 9).

Rate sensitivity of a present value

For a single payment due in T years, the sensitivity depends on the compounding convention. At a continuously compounded rate r , $PV = e^{-rT}$ per unit of payment and $dPV/dr = -T \cdot PV$: the relative sensitivity is exactly T . At an annually compounded rate y , $PV = (1 + y)^{-T}$ and $dPV/dy = -\frac{T}{1+y}PV$. For NT\$1,000,000 due in 5 years at 2%, the sensitivities are -452.42 and -443.99 per bp, on present values of 904,837.42 and 905,730.81.

The same reasoning applies to the index futures fair value $F = S e^{(r_f - q)\tau}$, which LM6 derives by no-arbitrage. Holding r_f , q and τ fixed, $dF/dS = e^{(r_f - q)\tau}$, close to 1.

Holding S , r_f and τ fixed, $dF/dq = -\tau F$. For a three-month contract ($\tau = 0.25$) at $F = 22,000$, a rise of 0.1 percentage point in the expected dividend yield lowers fair value by 5.50 points. A derivative that holds the other inputs fixed is a *partial* derivative, treated properly in LM7.

KNOWLEDGE CHECK 4

A trader is long 3 MTX and short 1 TX. What is the position's delta, and what is its profit if the futures price falls 120 points?

ANSWER

Delta is $3 \times 50 - 200 = -50$ NT\$ per point: the position is net short. A 120-point fall earns $(-50) \times (-120) = \text{NT\$}6,000$.

6 ELASTICITY AND LEVERAGE

- ☐ calculate elasticities, and show that the elasticity of a futures account's equity with respect to the futures price equals its leverage, exposure $\div$ equity

A unit-free derivative

A delta of NT\$200 per point cannot be compared with a duration of 9 years, and neither says what happens to an account in percent. Elasticity removes units by comparing *relative* changes.

DEFINITION Elasticity

For a positive function h of a positive variable x , the **elasticity** of h with respect to x is

$$\eta_{h,x} := \frac{dh/h}{dx/x} = \frac{x h'(x)}{h(x)} = \frac{d \ln h}{d \ln x}. \quad (2.7)$$

It is the percentage change in h per 1% change in x , to first order.

The last form follows from the chain rule. Put $z = \ln x$, so $x = e^z$ and $\ln h = \ln h(e^z)$. Then $\frac{d \ln h}{dz} = \frac{h'(e^z)e^z}{h(e^z)} = \frac{x h'(x)}{h(x)}$.

Because elasticities are derivatives of logarithms, the rules of Section 4 become rules of addition and multiplication. For positive functions u and h :

- **power:** if h is proportional to x^α , then $\eta_{h,x} = \alpha$;
- **product and quotient:** $\eta_{uh} = \eta_u + \eta_h$ and $\eta_{u/h} = \eta_u - \eta_h$;
- **chain:** the elasticity of $h(u(x))$ with respect to x is $\eta_{h,u} \cdot \eta_{u,x}$;
- **sum:** $\eta_{u+h} = \frac{u}{u+h} \eta_u + \frac{h}{u+h} \eta_h$, a value-weighted average.

The first three follow from $\ln(uh) = \ln u + \ln h$, $\ln x^\alpha = \alpha \ln x$ and (2.5). For the sum, $\eta_{u+h} = \frac{x(u'+h')}{u+h} = \frac{u}{u+h} \cdot \frac{xu'}{u} + \frac{h}{u+h} \cdot \frac{xh'}{h}$.

The sum rule is why portfolio sensitivities aggregate by value weights. A portfolio 60% in an index fund (elasticity 1) and 40% in cash (elasticity 0) has elasticity 0.6. Exhibit 6 gives each rule a financial reading.

Exhibit 6: Elasticity rules and their financial readings

Relation	Elasticity	Financial reading
$h \propto x^\alpha$	α	zero-coupon price $100(1+y)^{-T}$: elasticity $-T$ in $1+y$
uh	$\eta_u + \eta_h$	TWD value of a USD asset = USD value $\times$ exchange rate
u/h	$\eta_u - \eta_h$	$f = NmF/E$ has elasticity $1 - \eta_{E,F}$
$h(u(x))$	$\eta_{h,u} \eta_{u,x}$	account $\rightarrow$ futures $\rightarrow$ index
$u+h$	value-weighted average	portfolio sensitivity

Note: for a coupon bond, (2.6) gives $(1+y)P'(y)/P = -D_{\text{Mac}}$. Macaulay duration is minus the elasticity of price with respect to the gross yield $1+y$.

The elasticity of a futures account is its leverage

An account holding N contracts has equity $E(F) = E_0 + Nm(F - F_0)$, so $E'(F) = Nm$. By (2.7),

$$\eta_{E,F} = \frac{F \cdot Nm}{E} = \frac{NmF}{E} = f. \quad (2.8)$$

The elasticity of equity with respect to the futures price is the notional divided by equity: the account's leverage. Because E is linear in F , the one-day relation is exact, not approximate: $\Delta E/E = Nm \Delta F/E = f \cdot \Delta F/F$.

The index enters through the chain rule. With the cost-of-carry price $F = S e^{(r_f - q)\tau}$ and r_f, q, τ fixed, $\eta_{F,S} = S e^{(r_f - q)\tau} / F = 1$. Therefore $\eta_{E,S} = \eta_{E,F} \eta_{F,S} = f$.

RESULT Equity elasticity equals leverage

For a futures account, $\eta_{E,F} = f = NmF/E$ exactly, and under cost of carry $\eta_{E,S} = f$. The same identity holds for any position financed with fixed debt. If a position worth V is funded by equity E and debt $V - E$, then $\eta_{E,V} = V/E = f$.

This covers margin purchases. Buy 1,000 shares of 0050 at an illustrative NT\$60 with NT\$24,000 of your own and NT\$36,000 borrowed: $f = 60,000/24,000 = 2.50$, so a 4% fall removes 10% of equity.

EXAMPLE 7

Resolving the motivating case

Use (2.8) and Exhibit 1 to explain both traders' returns on 11 June and on 7 April 2025. Then explain the ratio of each account's return to TAIEX's.

SOLUTION

On 11 June the notional at the previous settlement was $200 \times 22,116 = \text{NT}\$4,423,200$. So $f = 4,423,200/600,000 = 7.3720$ for A and $4,423,200/900,000 = 4.9147$ for B. The futures return was $+1.0309\%$, so A earned $7.3720 \times 1.0309\% = 7.600\%$ and B earned $4.9147 \times 1.0309\% = 5.067\%$, exactly as observed.

Against TAIEX's $+1.0249\%$, the ratios are $7.600/1.0249 = 7.415$ and $5.067/1.0249 = 4.944$. They differ slightly from f because the futures and the index moved by different percentages. The ratio of their returns, a measured $\eta_{F,S}$, was $1.0309/1.0249 = 1.0059$. On 7 April it was 1.0307 , and the same arithmetic gives $7.0987 \times (-9.9972\%) = -70.967\%$ for A and $4.7324 \times (-9.9972\%) = -47.311\%$ for B.

Over the 145 sessions from 2 April to 31 October 2025, a least-squares line through daily TX and TAIEX returns has slope 1.008 (correlation 0.978). Day by day the ratio scatters: on the 14 days when TAIEX moved between 0.9% and 1.1% in size, it ranged from 0.753 to 1.459 .

TAIEX closes on a 13:25–13:30 call auction; the TX settlement comes from the last minute of trading to 13:45. The basis also moves with dividend expectations. On average the measured $\eta_{F,S}$ is close to 1; on any one day it is noisy.

The answer to "why 5–8%?" is therefore not the contract but the equity behind it: the multiple is notional $\div$ equity. With the illustrative initial margin of $\text{NT}\$350,000$ as the only equity, f on 11 June would have been 12.64 .

The exchange's margins cap leverage. TAIFEX's table effective 12 August 2026 requires an initial margin of $\text{NT}\$701,000$ and a maintenance margin of $\text{NT}\$538,000$ per TX contract. The MTX and TMF figures, $\text{NT}\$175,250$ and $\text{NT}\$35,050$ initial, are exactly $1/4$ and $1/20$ of TX, in proportion to the multipliers.

TAIFEX recalculates margin levels after each regular session and may adjust them when the change reaches a set threshold. That is why this module uses an illustrative 2025 figure rather than a quoted one. Scaled by TAIFEX's ratio $538,000/701,000$, the illustrative initial margin of $\text{NT}\$350,000$ corresponds to an illustrative maintenance margin of about $\text{NT}\$268,600$.

Leverage moves: its elasticity is $1 - f$

Leverage is a ratio, so its elasticity follows from the quotient rule for elasticities and (2.8):

$$\eta_{f,F} = \eta_{F,F} - \eta_{E,F} = 1 - f. \quad (2.9)$$

This is the elasticity form of Example 4. For $f > 1$ it is negative.

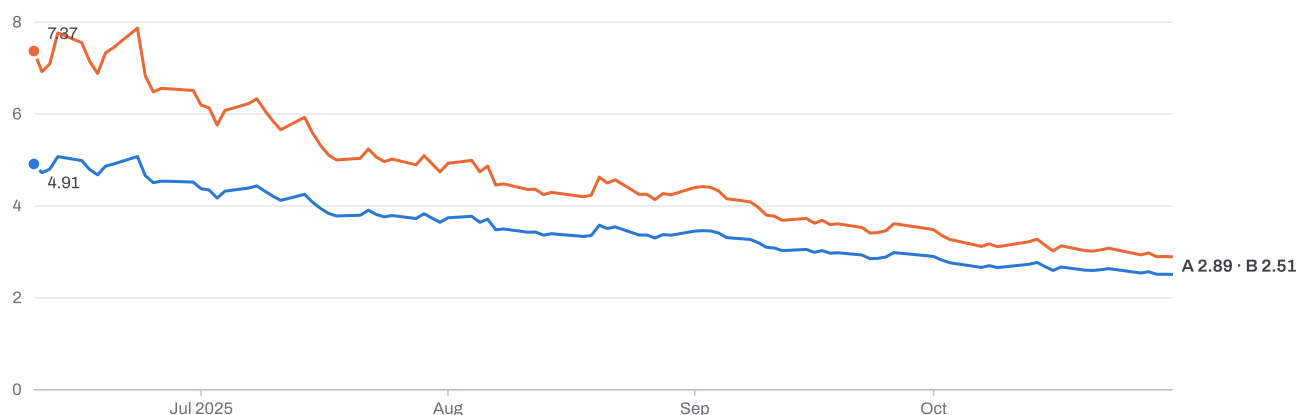
A 1% rise in the futures price cuts Trader A's leverage of 7.3720 by about 6.4% , to a first-order 6.9023 . The exact value is 6.9345 . On 11 June itself, a 1.0309% rise took A from 7.3720 to 6.9219 , against a linear prediction of 6.8877 .

Over months the drift is large. Exhibit 7 follows both accounts from the close of 10 June to 31 October 2025, holding one near-month contract and rolling the day before each expiry. TAIEX rose 26.94% and the futures position gained 6,824 points, or NT\$1,364,800. A's leverage fell from 7.37 to 2.89 and B's from 4.91 to 2.51.

Exhibit 7: Leverage of one TX contract held from 10 June to 31 October 2025

— Account A: NT\$600,000 on 10 June — Account B: NT\$900,000 on 10 June

Leverage $f = \text{notional} \div \text{equity}$



Note: one near-month TX contract, rolled at the settlement price on the day before each expiry, marked to market daily at NT\$200 per point; no deposits or withdrawals. On each roll day the new contract traded at least 35,485 contracts. Leverage $f = 200F/E$ at each settlement. Source: FinMind [TaiwanFuturesDaily](#), retrieved 10 October 2026; computations in [data/lm02_case.py](#).

Despite falling leverage, A's equity rose 227.47% and B's 151.64%. A fixed position's profit is linear in index points. Its return on the *initial* equity is therefore f_0 times the points gained divided by F_0 : $7.3720 \times 6,824/22,116 = 227.47\%$. Multiplying f_0 by TAIEX's return would have predicted only $7.3720 \times 26.94\% \approx 198.6\%$.

The futures also gained 832.79 more points than the index (6,824 against 5,991.21). The rolls added a net 510 points: +565 from the June, July and August rolls into cheaper contracts, -55 from the September and October rolls. The other 322.79 points came from the basis, which moved from a 126.14-point discount on 10 June to a 196.65-point premium on 31 October. LM6 explains this basis.

PITFALL "My leverage is 7" is a statement about today

Leverage, like any elasticity, is local. A fixed number of contracts deleverages in a rally and leverages up in a sell-off, at the rate $1 - f$ of (2.9). A daily-reset fund with multiple L does the opposite: it trades every day to hold $f = L$, which produces the volatility drag of LM3. Neither is "the" leverage, so a risk limit of "at most $f = 5$ " must say how often the position is rebalanced back to 5.

EXAMPLE 8

The measured elasticity of 00631L to 0050

00631L is designed to return twice the daily return of the Taiwan 50 index ($L = 2$), which 0050 tracks. Using daily total returns from 3 November 2014 to 8 October 2026, estimate its elasticity with respect to 0050, and comment.

SOLUTION

To first order in R_{0050} , $R_{00631L} \approx \eta R_{0050}$. The slope of a least-squares line through the 2,902 pairs of daily returns therefore estimates an average daily elasticity (LM23 treats regression).

Each fund split after a trading halt: 0050 4-for-1 on 18 June 2025 and 00631L 22-for-1 on 31 March 2026. The previous close is restated per new share, and returns run between common trading dates, so two pairs span the halts (six and five sessions). Unadjusted closes would give a slope of 0.846.

The slope is 1.887 and the correlation 0.974, with or without the two halt pairs. By full calendar year it ranged from 1.717 (2024) to 1.984 (2021).

On 7 April 2025 both funds closed at their daily price limits: 0050 fell 10.000% and 00631L 19.995%. TWSE sets the limit of a leveraged ETF on a domestic index at 10% times its multiple, so 00631L's is 20%. That day's ratio of 2.000 therefore reflects the exchange's limits, not the fund. The next session 0050 fell 3.25% and 00631L 11.20%, a ratio of 3.45.

The designed elasticity is 2; the measured one is lower on average. One likely source is replication basis, which LM3 discusses for the fund's long-run returns: the fund holds mostly TAIEX futures rather than Taiwan 50 instruments. Its market price can also drift from its net asset value. A designed elasticity still has to be measured.

EXAMPLE 9

A TWD investor in a USD asset

A Taiwan investor holds a US equity fund. Over a quarter the fund returns $R^{\text{USD}} = +6.00\%$ in USD, while the exchange rate X , TWD per USD, returns $R^{\text{FX}} = -6.00\%$ (illustrative numbers). Compute the investor's TWD return exactly and to first order, and describe what a currency hedge changes.

SOLUTION

The TWD value is $V = V^{\text{USD}} \times X$, so by the product rule its elasticity with respect to each factor is 1. Log returns add exactly, as in (1.15) of LM1: $r^{\text{TWD}} = \ln 1.06 + \ln 0.94 = -0.00361$, a simple return of $1.06 \times 0.94 - 1 = -0.36\%$.

The first-order answer, $6\% - 6\% = 0$, misses the second-order cross term $R^{\text{USD}} R^{\text{FX}} = -0.36\%$. Selling the initial USD value forward replaces most of the currency return with the forward premium (LM6). The hedge is not exact: the 6% gain in USD stays exposed to the exchange rate.

導讀 彈性 = 槓桿倍數

彈性是「百分比對百分比」的導數，沒有單位，所以能比較台指期、債券與外幣資產。期貨帳戶權益數對期貨價格的彈性，恰好等於名目價值（契約價值） $\div$ 權益數 $= f$ ：同一口台指期，掛在 60 萬權益數上是 7.37 倍，掛在 90 萬上只有 4.91 倍，所以 1% 的行情變成 7.6% 或 5.1%。關鍵是 f 本身也會動： $\eta_{f,F} = 1 - f$ ，固定口數時，漲了自動降槓桿，跌了槓桿暴增。每日重設的槓桿 ETF 反過來每天調整部位維持固定倍數 L ，這就是 LM3 波動拖累 (volatility drag) 的來源。

KNOWLEDGE CHECK 5

An account holds one TX contract with $f = 6$. Estimate its leverage after a 1% rise in the futures price, then compute it exactly.

ANSWER

By (2.9) the elasticity is $1 - 6 = -5$, so f falls by about 5%, to $6 \times 0.95 = 5.70$. Exactly, the new notional is 1.01 times the old and equity is $1 + 6 \times 0.01 = 1.06$ times the old, so $f = 6 \times 1.01/1.06 = 5.7170$.

7

DIFFERENTIALS AND LINEAR APPROXIMATION

- ☐ estimate changes with differentials, bound the error of a linear approximation, and explain when it fails

The differential

A derivative turns into an approximation by running the definition backwards. If Δx is small, $h(x_0 + \Delta x) - h(x_0)$ is close to $h'(x_0) \Delta x$.

DEFINITION Differential and linear approximation

If h is differentiable at x_0 , the **differential** $dh = h'(x_0) dx$ is the change along the tangent line for an input change dx . The **linear approximation** of h at x_0 is $T_1(x) := h(x_0) + h'(x_0)(x - x_0)$, the first-order Taylor polynomial of LM3.

By the definition of the derivative, the error $\text{err}(x) := h(x) - T_1(x)$ satisfies $\text{err}(x)/(x - x_0) \rightarrow 0$: it vanishes faster than the step. When h has a bounded second derivative, the mean value theorem makes this quantitative.

RESULT Size of the linearization error

If h is twice differentiable between x_0 and x , with $|h''| \leq M$ there, then

$$|h(x) - T_1(x)| \leq M(x - x_0)^2. \quad (2.10)$$

The error of a linear approximation is $O((x - x_0)^2)$: halve the step and the bound falls by a factor of 4.

Proof. By the mean value theorem, $h(x) - h(x_0) = h'(\xi)(x - x_0)$ for some ξ between x_0 and x . Hence $\text{err}(x) = [h'(\xi) - h'(x_0)](x - x_0)$. Applying the theorem to h' gives $h'(\xi) - h'(x_0) = h''(\zeta)(\xi - x_0)$, and $|\xi - x_0| \leq |x - x_0|$. ■ LM3 sharpens the constant from M to $M/2$ and gives the error exactly.

Linearizing e^x at 0 gives errors $e^x - (1 + x)$ of 0.021403, 0.005171 and 0.001271 for $x = 0.2, 0.1$ and 0.05 . Each halving divides the error by about four (4.14, then 4.07).

When the linear approximation fails

The bound (2.10) names the two enemies: a large step $|x - x_0|$ and a large curvature M . Four situations defeat a tangent line.

1. **Large moves.** The error grows like the square of the move. A tangent fitted at today's yield or price is a local tool.
2. **Singularities.** Near a point where h blows up, M is huge and no tangent follows the curve. Leverage near the zero-equity level is the example below.
3. **Kinks and jumps.** At a corner, such as a call payoff at the strike, no tangent exists. Across a jump, such as a gap or a limit-locked session, no continuous model applies.
4. **Stationary points.** If $h'(x_0) = 0$, the tangent is flat and says nothing about the direction of change. The second derivative decides, which is the business of LM4 (optima) and LM3 (convexity).

April 2025 shows the second case in real data. Trader B held one April contract on NT\$900,000 at the close of 2 April, with $f = 4.732$. Equity reaches zero at $F_z = F_0 - E_0/(Nm) = 21,296 - 900,000/200 = 16,796$, where f has a vertical asymptote. Exhibit 8 compares B's exact leverage with the linear approximation $f(F_0) + f'(F_0)(F - F_0)$ built on 2 April from Example 4.

Exhibit 8: Leverage of a one-contract account in April 2025: exact and linear

Date	TX settlement	Move since 2 April	Equity B (NT\$)	Leverage B, exact	Leverage B, linear
2 April	21,296	—	900,000	4.732	4.732
7 April	19,167	−10.00%	474,200	8.084	6.498
8 April	18,169	−14.68%	274,600	13.233	7.326
9 April	17,184	−19.31%	77,600	44.289	8.143
10 April	18,902	−11.24%	421,200	8.975	6.718

Note: one April TX contract held from the 2 April settlement with NT\$900,000 of equity; no margin calls, deposits or withdrawals are modelled. Trader A, with NT\$600,000, would have had NT\$174,200 of equity ($f = 22.006$) after 7 April and negative equity (−NT\$25,400) after 8 April: its zero-equity level was 18,296. Source: FinMind

TaiwanFuturesDaily, retrieved 10 October 2026.

For a hypothetical 1% fall from 2 April, the linear estimate (4.9091) misses the exact value (4.9179) by 0.0088. For the 19.31% fall to 9 April it predicts 8.143 when the truth is 44.289. The derivative at 2 April was correct; it simply stopped describing the function. Near F_z , f behaves like $1/(F - F_z)$, whose curvature explodes.

In practice neither account would have stayed untouched. A margin call comes when equity falls below the maintenance margin, illustratively NT\$268,600 here. A's equity fell below it after 7 April (NT\$174,200). B stayed above it after 8 April (NT\$274,600) and fell far below it after 9 April (NT\$77,600); either trader then had to deposit more or close the position.

導讀 線性近似什麼時候會失效

線性近似（切線）的誤差和「步長平方 × 曲率」同階，所以有四種情況會失效：變動太大、靠近奇異點（曲率爆炸）、尖角或跳空（根本沒有切線）、以及導數剛好為 0 的駐點。2025 年 4 月的台指期就是第二種：B 帳戶 4/2 槓桿 4.73 倍，三個交易日後期貨跌 19.3%，用切線估計只會到 8.1 倍，實際卻是 44.3 倍，因為權益數逼近 0，槓桿在「權益歸零點」16,796 有垂直漸近線。結論：導數是「當下」的敏感度，大行情時要重新計算，不能拿昨天的 delta 或 duration 線性外推。

KNOWLEDGE CHECK 6

Use a differential to estimate the price change, per 100 face, of the 10-year 2% bond when its yield rises from 2.00% to 2.05%.

ANSWER

$dP = P'(0.02) dy = -898.2585 \times 0.0005 = -0.4491$. Exact repricing gives -0.4480 ; the difference is the second-order convexity effect of LM3.

8

NUMERICAL DIFFERENTIATION

- ☐ compute derivatives numerically with forward and central differences, and choose a step size that balances truncation and rounding error

Many pricing functions are code with no usable derivative formula: a callable bond on a tree, a Monte Carlo option value, an interpolated curve. Practitioners then **bump and reprice**: they evaluate the function at nearby inputs and form a difference quotient. Doing this well requires knowing two errors that pull in opposite directions.

Forward and central differences

For a step $\Delta x > 0$, the **forward**, **backward** and **central** difference quotients are

$$\begin{aligned} Q_+(\Delta x) &= \frac{h(x + \Delta x) - h(x)}{\Delta x}, & Q_-(\Delta x) &= \frac{h(x) - h(x - \Delta x)}{\Delta x}, \\ Q_c(\Delta x) &= \frac{h(x + \Delta x) - h(x - \Delta x)}{2\Delta x}. \end{aligned} \tag{2.11}$$

The central quotient is the average of the other two.

Truncation error of the forward quotient. By the mean value theorem, $Q_+(\Delta x) = h'(\xi)$ for some $\xi \in (x, x + \Delta x)$. Applying the theorem again to h' , $Q_+(\Delta x) - h'(x) = h'(\xi) - h'(x) = h''(\zeta)(\xi - x)$. If $|h''| \leq M_2$ near x , then $|Q_+(\Delta x) - h'(x)| \leq M_2 \Delta x$: the error is $O(\Delta x)$.

Truncation error of the central quotient. If $|h'''| \leq M_3$ near x , then $|Q_c(\Delta x) - h'(x)| \leq M_3 \Delta x^2$: the error is $O(\Delta x^2)$. The odd symmetry of the central quotient cancels the first-order error term. The Deep Dive proves it with the mean value theorem alone.

DEEP DIVE Why the central difference is second order optional · click to expand

Let $J(\theta) = h(x + \theta) - h(x - \theta)$, so $J(0) = 0$ and $Q_c(\Delta x) = J(\Delta x)/(2\Delta x)$. By the mean value theorem, $J(\Delta x) = \Delta x J'(\theta)$ for some $\theta \in (0, \Delta x)$, and $J'(\theta) = h'(x + \theta) + h'(x - \theta)$. Therefore

$$Q_c(\Delta x) - h'(x) = \frac{1}{2} \left[(h'(x + \theta) - h'(x)) - (h'(x) - h'(x - \theta)) \right] = \frac{\theta}{2} [h''(\xi_1) - h''(\xi_2)]$$

with $\xi_1 \in (x, x + \theta)$ and $\xi_2 \in (x - \theta, x)$, using the theorem on h' twice. Once more, $|h''(\xi_1) - h''(\xi_2)| \leq M_3 |\xi_1 - \xi_2| \leq 2\theta M_3$, hence $|Q_c(\Delta x) - h'(x)| \leq \theta^2 M_3 \leq M_3 \Delta x^2$. ■

Taylor's theorem in LM3 sharpens the constants: the forward error is $\frac{1}{2}h''(x)\Delta x$ and the central error $\frac{1}{6}h'''(x)\Delta x^2$, up to higher-order terms.

The orders are visible in the numbers. For $h(x) = e^x$ at 0, the forward error divided by Δx is 0.5171, 0.5084 and 0.5042 for $\Delta x = 0.1, 0.05$ and 0.025 , approaching $h''(0)/2 = 0.5$. The central error divided by Δx^2 is 0.16675, 0.16669 and 0.16667, approaching $h'''(0)/6 = 1/6$.

Rounding error and the best step

Double precision stores about 16 significant digits. The gap between 1 and the next representable number is **machine epsilon**, $\varepsilon_m = 2.22 \times 10^{-16}$, so each computed $h(x \pm \Delta x)$ carries a rounding error of order $\varepsilon_m |h(x)|$. Subtracting two nearly equal values keeps those errors while the true difference shrinks, and dividing by Δx magnifies them.

The total error of the forward quotient is therefore about

$$\text{err}_+(\Delta x) \approx \frac{1}{2}|h''| \Delta x + \frac{2\varepsilon_m |h|}{\Delta x}, \quad \text{minimized at } \Delta x_+^* = 2\sqrt{\varepsilon_m |h|/|h''|},$$

and that of the central quotient about

$$\text{err}_c(\Delta x) \approx \frac{1}{6}|h'''| \Delta x^2 + \frac{\varepsilon_m |h|}{\Delta x}, \quad \text{minimized at } \Delta x_c^* = (3\varepsilon_m |h|/|h'''|)^{1/3}.$$

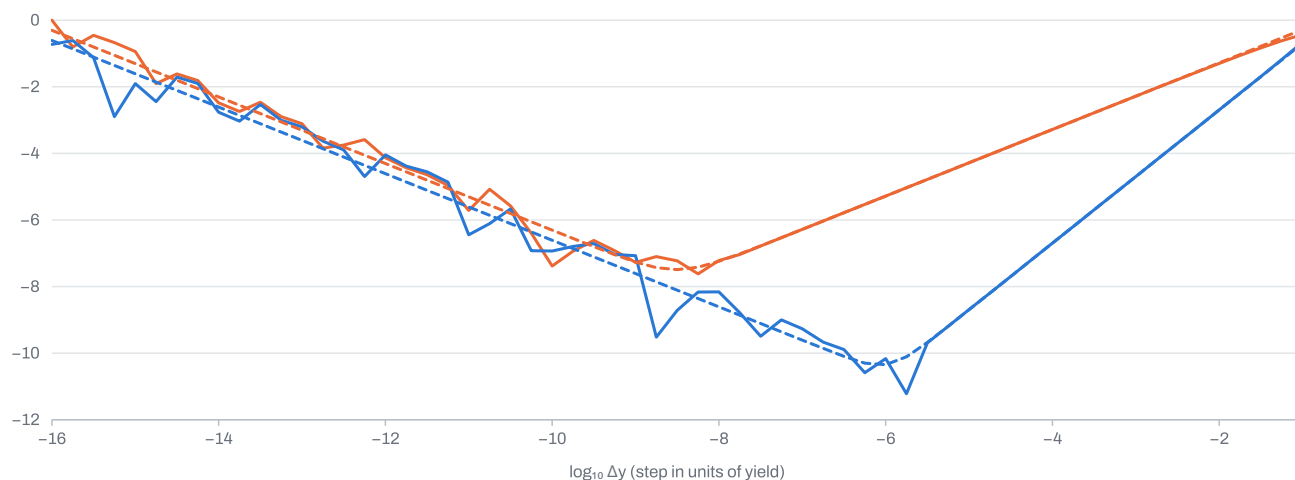
These are order-of-magnitude models, with the truncation constants taken from LM3. The lessons are robust: the best forward step scales like $\sqrt{\varepsilon_m} = 1.5 \times 10^{-8}$, the best central step like $\varepsilon_m^{1/3} = 6.1 \times 10^{-6}$, each times the scale of the problem.

For the bond, $|P| = 100$, $P''(2\%) = 9,399.5462$ and $P'''(2\%) = -108,918.1833$. The models give $\Delta y_+^* \approx 3 \times 10^{-9}$ and $\Delta y_c^* \approx 8 \times 10^{-7}$. Exhibit 9 scans Δy from 10^{-16} to 10^{-1} ; the smallest errors occur at $\Delta y = 10^{-8.25}$ (forward, relative error 2.4×10^{-8}) and $\Delta y = 10^{-5.75}$ (central, relative error 6.1×10^{-12}).

Exhibit 9: Error of numerical derivatives of the bond price versus step size

— Forward difference — Central difference - - - Model: $\frac{1}{2}|P''|\Delta y + 2\varepsilon_m|P|/\Delta y$ - - - Model: $|P'''|\Delta y^2/6 + \varepsilon_m|P|/\Delta y$

$\log_{10}$ of relative error in $P'(2\%)$



Note: relative error of $Q_+(\Delta y)$ and $Q_c(\Delta y)$ as estimates of $P'(2\%) = -898.2585$ for the 10-year 2% bond, both axes in powers of ten. On the right, truncation dominates and the lines have slopes 1 and 2. On the left, rounding dominates and the error grows like $1/\Delta y$. Dashed lines: the two error models above.

Exhibit 10: Forward and central estimates of $P'(2\%)$ at selected steps

Step Δy	Forward $Q_+(\Delta y)$	Relative error	Central $Q_c(\Delta y)$	Relative error
10^{-2}	-853.02028368	$+5.0 \times 10^{-2}$	-900.07536837	-2.0×10^{-3}
10^{-4}	-897.78870479	$+5.2 \times 10^{-4}$	-898.25868215	-2.0×10^{-7}
10^{-6}	-898.25380077	$+5.2 \times 10^{-6}$	-898.25850056	$+6.9 \times 10^{-11}$
10^{-8}	-898.25844611	$+6.1 \times 10^{-8}$	-898.25849443	$+6.9 \times 10^{-9}$
10^{-10}	-898.25846317	$+4.2 \times 10^{-8}$	-898.25860528	-1.2×10^{-7}
10^{-12}	-898.32496997	-7.4×10^{-5}	-898.33918082	-9.0×10^{-5}

Note: exact value $P'(2\%) = -898.25850062$. Below $\Delta y = 10^{-6}$ the central quotient loses accuracy to rounding. At 10^{-12} both quotients are wrong in the fourth significant digit, worse than a forward quotient with $\Delta y = 10^{-6}$.

EXAMPLE 10

Duration by bumping the yield

Estimate the modified duration of the 10-year bond by bumping its yield 1 bp, first with a forward and then with a central difference. Compare both with the exact value.

SOLUTION

With $\Delta y = 0.0001$, the forward estimate is $-Q_+(\Delta y)/P = 8.977887$, an error of -0.004698 . The central estimate is $-Q_c(\Delta y)/P = 8.9825868$, against the exact 8.9825850 : an error of $+1.8 \times 10^{-6}$. The central bump is about 2,600 times more accurate at the cost of one extra pricing.

This is why effective duration, $[P(y - \Delta y) - P(y + \Delta y)]/[2 \Delta y P(y)]$ for a parallel shift Δy of the yield curve, is a central difference. A 1 bp step is far above the rounding floor, so it is safe for any well-behaved pricer.

PITFALL Smaller is not better, and kinks fool every formula

Shrinking the step below the optimum makes the estimate *worse*, as Exhibit 10 shows at $\Delta y = 10^{-12}$. At a kink no step helps. For the call payoff $\max(x - 1, 0)$ at the strike $x = 1$, the forward, backward and central quotients are 1, 0 and 0.5 for every Δx . The central value looks like a delta but is not a derivative; numerical Greeks near strikes and barriers need care (LM31).

導讀 數值微分：步長不是越小越好

用差分估計導數（實務上叫 bump and reprice）有兩種誤差在拉鋸：截斷誤差隨步長 Δx 變小而變小（前向差分 $O(\Delta x)$ 、中央差分 $O(\Delta x^2)$ ），捨入誤差卻隨 Δx 變小而變大（約 $\varepsilon_m/\Delta x$ ，雙精度 $\varepsilon_m = 2.22 \times 10^{-16}$ ）。所以最佳步長在中間：前向差分約 10^{-8} 、中央差分約 10^{-6} （乘上問題的尺度）。實務上算債券 duration 用 1 bp 的中央差分就很準：誤差只有 1.8×10^{-6} （相對誤差約 2×10^{-7} ）；但碰到買權履約價這種尖角，任何步長都算不出真正的導數。

KNOWLEDGE CHECK 7

Where truncation error dominates, by what factor does halving the step divide the forward error, and the central error? Check with $h(x) = \ln x$ at $x = 2$, using $\Delta x = 0.01$ and $\Delta x = 0.005$.

ANSWER

By about 2 and about 4. For $\ln x$ at 2 the forward errors are -0.0012458 and -0.0006240 (ratio 2.00); the central errors are $+4.167 \times 10^{-6}$ and $+1.042 \times 10^{-6}$ (ratio 4.00).

9

DERIVATIVES IN PYTHON

- implement and verify derivatives in Python with finite differences and symbolic differentiation

Three habits make derivatives safe in code. Compare every numerical derivative with an exact one on a case where both exist. Bump with central differences and a step well above the rounding floor. And let a computer algebra system check hand-derived formulas.

The first block implements (2.11) and reproduces Example 10:

PYTHON

```
def bond_price(y, coupon=0.02, n=10, face=100.0):
    """Annual-coupon bond priced at yield y; the 10-year 2% bond by default."""
    return (sum(coupon * face / (1 + y) ** t for t in range(1, n + 1))
            + face / (1 + y) ** n)

def forward_diff(fun, x, dx):
    return (fun(x + dx) - fun(x)) / dx

def central_diff(fun, x, dx):
    return (fun(x + dx) - fun(x - dx)) / (2 * dx)

y0, dy = 0.02, 1e-4
P0 = bond_price(y0)
print(f"D_mod forward {-forward_diff(bond_price, y0, dy) / P0:.6f}")  # 8.977887
print(f"D_mod central {-central_diff(bond_price, y0, dy) / P0:.6f}")  # 8.982587
```

SymPy differentiates the same function symbolically and checks the leverage derivative of Example 4:

PYTHON

```
import sympy as sp

y = sp.symbols("y", positive=True)
P = (sum(sp.Integer(2) / (1 + y) ** t for t in range(1, 11))
     + 100 / (1 + y) ** 10)
dP = sp.diff(P, y)  # exact derivative
print(sp.N(dP.subs(y, sp.Rational(1, 50)), 10))  # -898.2585006
print(sp.N((-dP / P).subs(y, sp.Rational(1, 50)), 6))  # 8.98259

F, E0, F0, N, m = sp.symbols("F E_0 F_0 N m", positive=True)
f = N * m * F / (E0 + N * m * (F - F0))  # exposure / equity
print(sp.simplify(sp.diff(f, F) + f * (f - 1) / F))  # 0, so f'(F) = -f(f-1)/F
```

Elasticities can be estimated from data. The account of the motivating case takes a few lines of Polars, and the slope of Example 8 one call to NumPy:

PYTHON

```
import numpy as np
import polars as pl

# paths from the repository root
tx = pl.read_csv("data/raw/TX__2025-04-01__2025-10-31.csv")
june = (tx.filter(pl.col("contract_date") == 202506)
        .filter(pl.col("date").is_in(["2025-06-10", "2025-06-11"]))
        .sort("date")["settlement_price"].to_numpy())
f_A = 200 * june[0] / 600_000 # exposure / equity
print(f"f_A = {f_A:.4f}, account return = "
      f"{f_A * (june[1] / june[0] - 1):.2%}") # 7.3720, 7.60%

# written by data/build_series.py
etf = pl.read_parquet("data/processed/etf_daily_returns.parquet")
slope = np.polyfit(etf["R_0050"].to_numpy(), etf["R_00631L"].to_numpy(), 1)[0]
print(f"elasticity of 00631L to 0050: {slope:.3f}") # 1.887
```

IN PYTHON Companion lab

The notebook `LM02_lab.ipynb` rebuilds Exhibit 1, Exhibit 2, Exhibit 3, Exhibit 7, Exhibit 8, Exhibit 9 and Exhibit 10. It runs finite-difference experiments on a log-log grid and computes duration by bumping yields. It also estimates the elasticity of levered positions from 0050 data and solves the computational practice problems. The notebook reads FinMind files that you download once with your own token. From the repository root, run `uv run python data/fetch_finmind.py`, then `uv run python data/make_lab_data.py`. FinMind's licence does not allow the book to redistribute the data. After that, the notebook runs offline.

SUMMARY

- A limit describes where a function is heading. From $1 + x \leq e^x \leq 1/(1 - x)$ and the squeeze theorem, $(e^x - 1)/x \rightarrow 1$ and $\ln(1 + x)/x \rightarrow 1$; these two limits drive every derivative of exponentials and logarithms.
- A function is continuous when its limit equals its value; discontinuities are removable, jumps (digital payoffs) or infinite (leverage at zero equity). A call payoff is continuous but not differentiable at the strike.
- The derivative (2.4) is a rate with units: a slope, a marginal rate and a sensitivity. Six rules proved from (2.4) differentiate every function built from powers, exponentials and logarithms.
- A futures position's delta is Nm NT\$ per point (TX 200, MTX 50, TMF 10). Modified duration is $-P'(y)/P$, and Macaulay duration emerges as the present-value-weighted time; the 10-year 2% par bond has $D_{\text{mod}} = 8.9826$.

- Elasticity is $d \ln h / d \ln x$. A futures account's equity elasticity equals its leverage $f = NmF/E$, exposure $\div$ equity, exactly for one mark-to-market period. On 11 June 2025, $f = 7.3720$ turned a 1.031% futures move into a 7.60% account move.
- The leverage of a fixed position has elasticity $1 - f$. Account A's leverage fell from 7.37 to 2.89 in the June–October 2025 rally; account B's rose from 4.73 to 44.29 in three April sessions.
- A linear approximation has error $O((x - x_0)^2)$ and fails for large moves, near singularities, at kinks and jumps, and at stationary points.
- Numerical derivatives trade truncation error ($O(\Delta x)$ forward, $O(\Delta x^2)$ central) against rounding error (of order $\varepsilon_m / \Delta x$). Best steps are near 10^{-8} and 10^{-6} times the problem's scale; a 1 bp central bump prices duration to within 0.000002.

PRACTICE PROBLEMS

1. $\lim_{x \rightarrow 0} (e^{2x} - 1)/x$ is:
 - A. 1
 - B. 2
 - C. e^2
2. The derivative of $h(x) = \ln(1 + 3x)$ at $x = 0$ is:
 - A. $1/3$
 - B. 1
 - C. 3
3. Use the definition (2.4) to find the derivative of $h(x) = \sqrt{x}$ at $x = 4$.
4. Which statement is correct?
 - A. Every differentiable function is continuous.
 - B. Every continuous function is differentiable.
 - C. The payoff $\max(S - K, 0)$ is differentiable at $S = K$.
5. A trader is long 1 TX, short 3 MTX and long 7 TMF in the same expiry month. Compute the position's delta and its profit if the futures price falls 250 points.
6. An account holds 2 MTX contracts at $F = 25,000$ with NT\$500,000 of equity. If the futures price rises 1.2%, the account's return is *closest* to:
 - A. 2.4%
 - B. 5.0%
 - C. 6.0%

7. A futures account has leverage $f = 8$. Using (2.9), estimate its leverage after the futures price rises 2% and after it falls 2%. Compute both exactly and explain why the linear estimate is worse for the fall.
8. A 5-year zero-coupon bond yields 2%. Compute its modified duration under annual compounding and its duration under continuous compounding.
9. A bond pays an annual coupon C per 100 face for n years and trades at par at its yield y_0 , so $C = 100 y_0$. Show that its modified duration is $D_{\text{mod}} = (1 - (1 + y_0)^{-n})/y_0$, the annuity factor $a_{n,i}$ at $i = y_0$. Check the formula for the 10-year 2% bond.
10. A Taiwan investor's USD fund returns $R^{\text{USD}} = +8\%$ over a year, while the exchange rate, TWD per USD, returns $R^{\text{FX}} = -5\%$. Compute the TWD return exactly and to first order, and identify the neglected term.
11. Estimate the derivative of $\ln x$ at $x = 2$ with forward and central differences using $\Delta x = 0.01$. Report each error.
12. In double precision, the step that minimizes the total error of a central difference for a function and argument of order 1 is *closest* to:
 - A. 10^{-16}
 - B. 10^{-6}
 - C. 10^{-8}
13. A bet returns +2% or -1% with equal probability, and a trader stakes the fraction f of capital on it. The expected log growth per bet is $g(f) = \frac{1}{2} \ln(1 + 0.02f) + \frac{1}{2} \ln(1 - 0.01f)$. Compute $g'(f)$, evaluate it at $f = 0, 10, 25$ and 40 , and interpret the signs.
14. The current yield of a bond with annual coupon C is $CY(y) = C/P(y)$. Show that $CY'(y) = CY \cdot D_{\text{mod}}$, and estimate the change in the 10-year 2% bond's current yield for a 10 bp rise in its yield.
15. Index futures fair value is $F = S e^{(r_f - q)\tau}$. Take $S = 22,470.10$, $r_f = 1.6\%$, $q = 3.0\%$ and $\tau = 0.25$ (illustrative inputs). Compute F , the elasticity of F with respect to S , and the change in F when q rises by 0.1 percentage point.
16. (Python) Compute forward and central estimates of the 10-year bond's $P'(2\%)$ for $\Delta y = 10^{-1}, 10^{-2}, \dots, 10^{-12}$. Which step is best for each method, and how do the answers compare with $\Delta y^* \approx \sqrt{\epsilon_m}$ and $\epsilon_m^{1/3}$?

SOLUTIONS

1. **B** is correct. With $z = 2x$, $(e^{2x} - 1)/x = 2(e^z - 1)/z \rightarrow 2$ by (2.2). A ignores the factor 2; C confuses the limit with e^{2x} at $x = 1$.
2. **C** is correct. By the chain rule, $h'(x) = 3/(1 + 3x)$, so $h'(0) = 3$. A inverts the inner derivative; B forgets it.
3. $\frac{\sqrt{4 + \Delta x} - 2}{\Delta x} = \frac{(4 + \Delta x) - 4}{\Delta x(\sqrt{4 + \Delta x} + 2)} = \frac{1}{\sqrt{4 + \Delta x} + 2} \rightarrow \frac{1}{4}$. So the derivative is 0.25, in agreement with $\frac{1}{2}x^{-1/2}$ at $x = 4$.
4. **A** is correct (Section 3). B fails for $|x|$ at 0. C fails because the left and right derivatives at K are 0 and 1.
5. Delta is $200 - 3 \times 50 + 7 \times 10 = 120$ NT\$ per point. A 250-point fall loses $120 \times 250 = \text{NT\$}30,000$.
6. **C** is correct. Notional is $2 \times 50 \times 25,000 = \text{NT\$}2,500,000$, so $f = 2,500,000/500,000 = 5.0$, and the return is $5.0 \times 1.2\% = 6.0\%$ exactly. A is twice the move (the number of contracts); B is the leverage itself.
7. By (2.9) the elasticity is $1 - 8 = -7$. Linear estimates: $8 \times (1 - 7 \times 0.02) = 6.88$ after the rise and $8 \times (1 + 7 \times 0.02) = 9.12$ after the fall. Exactly, $f = 8 \times 1.02/(1 + 8 \times 0.02) = 7.0345$ after the rise and $8 \times 0.98/(1 - 8 \times 0.02) = 9.3333$ after the fall. The fall moves the account toward zero equity, where f curves sharply upward (Section 7), so the tangent line falls further behind.
8. Annual compounding: $D_{\text{mod}} = 5/1.02 = 4.9020$. Continuous compounding: $dPV/dr = -5 PV$, so the duration is exactly 5.0000. The two differ by the factor $1 + y$: modified duration is Macaulay duration (here 5) divided by $1 + y$, and continuous compounding removes that factor.
9. Let $v = 1/(1 + y)$ and $a_{n,y} = (1 - v^n)/y$, so that $P(y) = C a_{n,y} + 100 v^n$. The chain rule gives $\frac{d}{dy}v^n = -n v^{n+1}$, and the quotient rule gives $\frac{d}{dy}a_{n,y} = [n y v^{n+1} - (1 - v^n)]/y^2$. At $y = y_0$, where $C = 100 y_0$:

$$P'(y_0) = 100 \left[\frac{n y_0 v^{n+1} - (1 - v^n)}{y_0} - n v^{n+1} \right] = -100 \frac{1 - v^n}{y_0} = -100 a_{n,y_0}.$$
 Since $P(y_0) = 100$ at par, $D_{\text{mod}} = -P'/P = a_{n,y_0}$. For $y_0 = 2\%$ and $n = 10$, $a_{n,y_0} = 8.9826$, matching Example 6.
10. Exactly, $1.08 \times 0.95 - 1 = +2.6000\%$. To first order, $8\% - 5\% = +3.00\%$. The neglected term is the cross product $R^{\text{USD}} R^{\text{FX}} = 0.08 \times (-0.05) = -0.4000\%$, which is second order; log returns add without it.
11. Forward: $(\ln 2.01 - \ln 2)/0.01 = 0.4987542$, error -0.0012458 . Central: $(\ln 2.01 - \ln 1.99)/0.02 = 0.500004167$, error $+4.167 \times 10^{-6}$. The exact value is 0.5; the central estimate is about 300 times more accurate.
12. **B** is correct. The best central step scales like $\varepsilon_m^{1/3} = 6.1 \times 10^{-6}$. C is the best *forward* step, $\sqrt{\varepsilon_m}$; A is machine epsilon itself, where rounding destroys the quotient.

13. $g'(f) = \frac{0.01}{1 + 0.02f} - \frac{0.005}{1 - 0.01f}$. Then $g'(0) = 0.00500$, the arithmetic mean return; $g'(10) = 0.002778$; $g'(25) = 0$; $g'(40) = -0.002778$. Growth rises with f up to $f = 25$ and falls beyond it, so $f^* = 25$ maximizes growth. LM4 develops this first-order condition and explains why such a large f should not be used as is.
14. By the quotient rule, $CY' = -C P'/P^2 = (C/P) \cdot (-P'/P) = CY \cdot D_{\text{mod}}$. For the 10-year bond, $CY = 2.0000\%$ and $CY' = 0.02 \times 8.9826 = 0.17965$, so a 10 bp rise lifts the current yield by about $0.17965 \times 0.001 = 1.80$ bp. Exact repricing, $2/P(2.1\%) = 2/99.1064$, gives 2.01803%, also +1.80 bp.
15. $F = 22,470.10 e^{(0.016 - 0.030) \times 0.25} = 22,391.6$. Since F is proportional to S , $\eta_{F,S} = 1$ exactly, although $dF/dS = e^{(r_f - q)\tau} = 0.996506$. $dF/dq = -\tau F = -5,597.9$ per unit of q , so a rise of 0.001 in q lowers F by 5.60 points.
16. The scan reproduces Exhibit 10. The central error is smallest at $\Delta y = 10^{-6}$ (relative error 6.9×10^{-11}). On this grid the forward error is smallest at 10^{-10} (4.2×10^{-8}), but 10^{-9} (5.3×10^{-8}) and 10^{-8} (6.1×10^{-8}) are within a factor of 1.5 of it: a plateau around the crossover of the two error terms at 3×10^{-9} .
- At 10^{-8} truncation still dominates (5.2×10^{-8} against 4.9×10^{-9} of rounding in the model). Below the crossover rounding dominates, and the low value at 10^{-10} (model 4.9×10^{-7}) is luck. The quarter-decade scan of Exhibit 9 puts the minima at $10^{-8.25}$ and $10^{-5.75}$, which match $\sqrt{\varepsilon_m} = 1.5 \times 10^{-8}$ and $\varepsilon_m^{1/3} = 6.1 \times 10^{-6}$ in order of magnitude.

GLOSSARY

Term	中文	Meaning
Limit	極限	Value ℓ approached by $h(x)$ as $x \rightarrow x_0$, regardless of $h(x_0)$.
One-sided limit	單邊極限	Limit as x approaches x_0 from the left or from the right only.
Squeeze theorem	夾擠定理	A function trapped between two functions with a common limit shares that limit.
Continuity	連續性	$\lim_{x \rightarrow x_0} h(x) = h(x_0)$.
Difference quotient	差商	$[h(x_0 + \Delta x) - h(x_0)]/\Delta x$, the slope of a secant line.
Derivative	導數	Limit of the difference quotient, (2.4); a rate with units.
Tangent line	切線	Line through $(x_0, h(x_0))$ with slope $h'(x_0)$.
Mean value theorem	均值定理	Some interior point has instantaneous rate equal to the average rate.
Chain rule	連鎖律	$\frac{d}{dx}h(u(x)) = h'(u(x))u'(x)$, (2.5).
Sensitivity	敏感度	Change in value per unit change in one risk factor, others fixed.
Contract multiplier	契約乘數	NT\$ value of one index point for one futures contract.
Notional value	名目價值 (契約價值)	Multiplier $\times$ futures price $\times$ number of contracts: the exposure.
Delta	Delta	Derivative of a position's value with respect to the underlying price.
Modified duration	修正存續期間	$-P'(y)/P$, (2.6).
Macaulay duration	麥考利存續期間	Present-value-weighted average time of cash flows; $(1 + y)D_{\text{mod}}$.
DV01	基點價值	Price fall for a one-basis-point rise in yield, $-P'(y) \times 0.0001$.
Elasticity	彈性	$d \ln h / d \ln x$, the percentage change in h per 1% change in x , (2.7).
Leverage	槓桿倍數	Exposure $\div$ equity, $f = NmF/E$; the equity elasticity of a futures account, (2.8).

Term	中文	Meaning
Initial margin	原始保證金	Equity required per contract to open a futures position.
Maintenance margin	維持保證金	Equity level below which a margin call is triggered.
Differential	微分	$dh = h'(x_0) dx$, the change along the tangent line.
Linear approximation	線性近似	$T_1(x) = h(x_0) + h'(x_0)(x - x_0)$; error $O((x - x_0)^2)$, (2.10).
Forward difference	前向差分	$[h(x + \Delta x) - h(x)]/\Delta x$; truncation error $O(\Delta x)$.
Central difference	中央差分	$[h(x + \Delta x) - h(x - \Delta x)]/(2\Delta x)$; truncation error $O(\Delta x^2)$.
Truncation error	截斷誤差	Error from replacing a limit by a finite step.
Rounding error	捨入誤差	Error from finite-precision arithmetic.
Machine epsilon	機器精度	Gap between 1 and the next double-precision number, 2.22×10^{-16} .

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