

Taylor Series and Approximation

How much is $\approx$ worth? Local polynomial models, their error, and the money they explain.

VOLUME I · Calculus for Finance

BUILDS ON LM1 · LM2

LEADS TO LM4 · LM5 · LM6 · LM7 · LM14 · LM28 · LM42 · LM46

TIME $\approx$ 2.5 h + lab

LEARNING OUTCOMES

Mastery

After this module you should be able to:

☐	1 construct the Taylor polynomial of order n of a smooth function about a point, and interpret its terms as level, slope and curvature
☐	2 derive the expansions of e^x , $\ln(1+x)$ and $(1+x)^\alpha$, and state where each one converges
☐	3 bound the truncation error with the Lagrange remainder, and decide when a truncated expansion can be trusted
☐	4 relate simple and log returns, and arithmetic and geometric means, through second-order expansions, and quantify volatility drag
☐	5 decompose the long-run return of a daily-reset leveraged ETF into a power of the index, a variance drag and a residual, and estimate each term from data
☐	6 apply first- and second-order approximations to bond prices (duration and convexity), and recognise the same structure in option delta–gamma and in Itô's lemma
☐	7 implement and verify Taylor approximations in Python

LEARNING MODULE OVERVIEW

- A Taylor polynomial is the best local polynomial model of a smooth function: it matches the function's value and its first n derivatives at one point.
- Three expansions — e^x , $\ln(1+x)$ and $(1+x)^\alpha$ — generate most approximations used in finance. The series for $\ln(1+x)$ converges only for $-1 < x \leq 1$.
- The Lagrange remainder turns “ $\approx$ ” into a statement with an error bound. The error of an order- n approximation shrinks like $|x|^{n+1}$.
- Log and simple returns differ by $R^2/2$ to second order. Over a sample, geometric return $\approx$ arithmetic mean – half the variance: this is volatility drag, and it is an approximation, not an identity.
- A daily-reset $L\times$ fund earns approximately the L -th power of the index's growth, taxed by $\exp\left(-\frac{L(L-1)}{2} \sum R_t^2\right)$. On twelve years of 0050 data the second-order term is accurate to a few thousandths of a log point. Over the full sample it comes within 0.0001, helped by cancelling third- and fourth-order terms.
- “Leveraged ETFs decay” and “leveraged ETFs beat $L\times$ ” are both true, against different benchmarks. In strong trends a $2\times$ fund can more than double the index's return.
- Duration–convexity, delta–gamma and Itô's lemma are all second-order Taylor expansions. Itô's lemma is the one in which the second-order term does not vanish.

1

INTRODUCTION

Almost every formula a trader uses daily is an approximation in disguise. Examples are “log return $\approx$ simple return”, “geometric mean $\approx$ arithmetic mean – $\sigma^2/2$ ” and “bond price return $\approx$ –duration $\times$ yield change”. Each is the first or second term of a **Taylor expansion**. This module shows where they come from, how wrong they are and when they break, and then uses them on real money.

MOTIVATING CASE Two numbers that cannot both be true

From the close of 31 October 2014 to 8 October 2026, Yuanta Taiwan 50 (0050) returned **+902%** including dividends: one unit grew to 10.02. Over the same days the daily-reset $2\times$ fund on that index, Yuanta Taiwan 50 Bull 2X (00631L), returned **+4,354%**: one unit grew to 44.54. “Twice the index's return” would have been +1,804%. The leveraged fund delivered about 2.4 times *twice* the index.

Then take calendar 2022 in the US. QQQ lost 32.6%; the $3\times$ fund TQQQ lost 79.1%. Three times QQQ's loss is –97.7%, so TQQQ lost *less* than three times the index.

Yet every regulator and textbook warns that leveraged ETFs **decay**. How can a fund decay and beat its multiple at the same time? How large is the decay, and what sets its size? By Section 7 you will split each fund's return into three numbers: a power of the index, a variance drag and a residual.

Exhibit 1 collects the numbers.

Exhibit 1: The leveraged-fund paradox in two markets			
Period	Index fund	Leveraged fund	“Multiple × index return”
2014-10-31 → 2026-10-08 (11.94 years)	0050: +902.25%	00631L (2×): +4,354.5%	$2 \times 902.25\% =$ +1,804.5%
Annualized	21.3%	37.4%	—
Calendar 2022	QQQ: −32.58%	TQQQ (3×): −79.09%	$3 \times (-32.58\%) = -97.7\%$

Source: FinMind `TaiwanStockPrice`, `TaiwanStockDividendResult` and `USStockPrice`, retrieved 10 October 2026. 0050 includes reinvested cash dividends and is adjusted for its 4-for-1 split (18 June 2025). 00631L is adjusted for its 22-for-1 split (31 March 2026); QQQ and TQQQ use adjusted closes. Every 0050 and 00631L figure in this module uses the 2,902 days on which both funds traded. They span 4,360 calendar days, so $A = 2,902 \times 365.25/4,360 = 243.1$ returns a year. 0050 was halted from 11 to 17 June 2025 for its split, and 00631L from 25 to 30 March 2026 for its own. The returns dated 18 June 2025 and 31 March 2026 therefore span six and five sessions. Computations: companion notebook `LM03_lab.ipynb`.

The tool that resolves the paradox is a second-order Taylor expansion of $\ln(1 + x)$. Sections 2–4 build the tool and its error bound. Sections 5–7 apply it to returns, means and leverage. Section 8 shows the same idea in bond pricing, option hedging and, as a preview of [LM28](#), Itô’s lemma.

NOTE **Notation**

As throughout the book, R is a **simple return** for one period, $R = P_t/P_{t-1} - 1$ (plus income), and $r = \ln(1 + R)$ is the corresponding **log return**. μ and σ are the mean and standard deviation of R in a model. $\bar{R}$ and s are their sample values, with s dividing by T . T counts returns, t indexes them (and serves as a variable of integration), and τ is a span in years.

g is a **log growth rate** per period and G a **growth factor** (end value ÷ start value). RV is realized variance and A the number of returns per year. L is a fund’s daily multiple, P a price, D_t a cash dividend, i a periodic interest rate and y a yield.

Local symbols: x is a generic real variable. In the purely mathematical Sections 2–4 and the problems, summary points and glossary entries on them, f is a generic smooth function of x . It is expanded about the point a , a generic constant, with Taylor polynomial $T_n(x)$ of order n , remainder $E_n(x)$ and intermediate point ξ . h is a general polynomial with coefficients c_k ; k indexes the terms of a series, m counts derivatives and α is an exponent.

Later sections add R_G (geometric mean return), π (inflation), j (an asset, with weight w_j) and G_{fund} (a fund’s growth factor). D_{mod} and C are modified duration and convexity. Section 8 alone uses an option’s value $V(S, t)$ on an underlying price S , with Greeks Δ , Γ and Θ . It also uses a standard normal shock Z , N steps and B_t , the book’s symbol for a standard Brownian motion.

Δ before a variable denotes a change. We write $=$ only for exact statements, $\approx$ for approximations whose neglected order is stated, and $:=$ for definitions.

- construct the Taylor polynomial of order n of a smooth function about a point, and interpret its terms as level, slope and curvature

Tangent line, then curvature

Near a point a , a smooth function looks like a straight line: its tangent. The tangent line is the first-order model

$$f(x) \approx f(a) + f'(a)(x - a).$$

For $f(x) = \ln(1 + x)$ at $a = 0$, $f(0) = 0$ and $f'(0) = 1$, so $\ln(1 + x) \approx x$. That is the familiar “log return $\approx$ simple return”. It is exact at $x = 0$ and drifts away as x grows, because $\ln(1 + x)$ bends downward. A better model adds the bend, measured by the second derivative:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

With $f''(x) = -1/(1 + x)^2$ and $f''(0) = -1$ this gives $\ln(1 + x) \approx x - x^2/2$. The $-x^2/2$ term is the curvature correction, and it is the source of every “ $-\sigma^2/2$ ” in this book.

DEFINITION Taylor polynomial

Let f have n derivatives at a . The **Taylor polynomial of order n of f about a** is

$$T_n(x) := \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n. \quad (3.1)$$

When $a = 0$ it is also called a **Maclaurin polynomial**. Letting $n \rightarrow \infty$ gives the **Taylor series**, when that series converges.

Why the coefficients are $f^{(k)}(a)/k!$

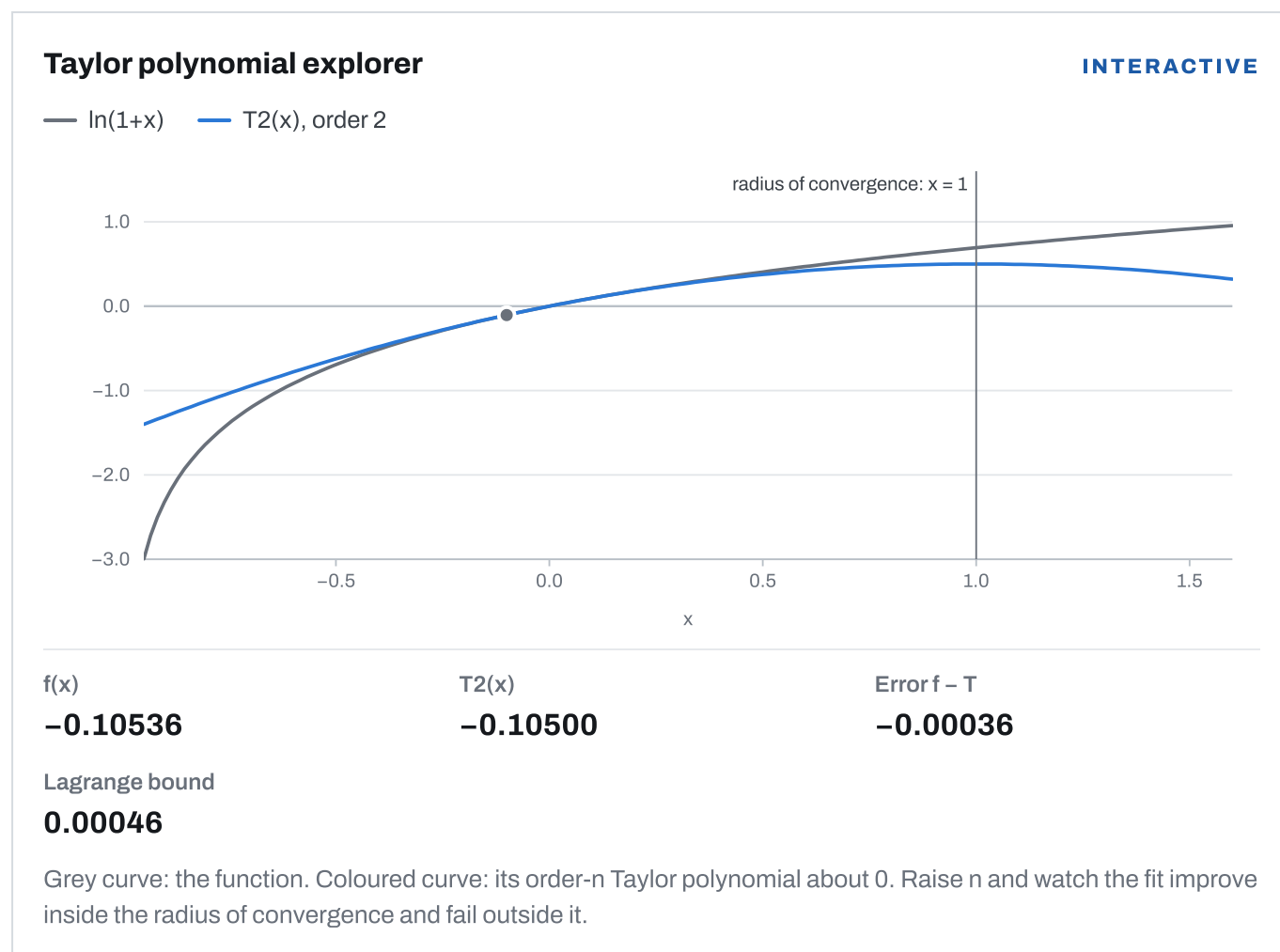
T_n is built so that it agrees with f at a in value, slope, curvature and every higher derivative up to order n . Write a general polynomial $h(x) = \sum_{k=0}^n c_k(x - a)^k$ and differentiate it m times. Every term with $k < m$ disappears, every term with $k > m$ still contains a factor $(x - a)$ and vanishes at $x = a$, and the term $k = m$ becomes $m! c_m$. So

$$h^{(m)}(a) = m! c_m.$$

Requiring $h^{(m)}(a) = f^{(m)}(a)$ for $m = 0, \dots, n$ forces $c_m = f^{(m)}(a)/m!$. The factorial is not decoration: it undoes the factorial that repeated differentiation of $(x - a)^m$ produces.

Each term has a job. The constant fixes the **level**, the linear term the **slope**, the quadratic term the **curvature**, and higher terms correct asymmetry and tails. In finance the terms through second order

usually suffice, which is why “level, slope, curvature” recurs in duration–convexity, delta–gamma and the volatility drag.



EXAMPLE 1

Approximating $\ln(1.05)$ and $e^{0.05}$

A stock returns +5% in a month. Approximate its log return $\ln(1.05)$ and the growth factor $e^{0.05}$ of a 5% continuously compounded rate. Use Taylor polynomials of order 1, 2 and 3 about 0, and report each error.

SOLUTION

For $\ln(1+x)$ with $x = 0.05$: $T_1 = 0.05$, $T_2 = 0.05 - 0.00125 = 0.04875$, $T_3 = 0.04875 + 0.0000417 = 0.0487917$. The exact value is $\ln 1.05 = 0.0487902$, so the errors $f - T_n$ are -0.0012098 , $+0.0000402$ and -0.0000015 .

For e^x : $T_1 = 1.05$, $T_2 = 1.05125$, $T_3 = 1.0512708$ against the exact 1.0512711 , with errors $+0.0012711$, $+0.0000211$ and $+0.0000003$.

The error after T_n is roughly the first omitted term, so each extra order shrinks it by the ratio of successive terms. For $\ln(1+x)$ that ratio is $x(n+1)/(n+2)$, here about $1/30$ and $1/27$. For e^x it is $x/(n+2)$, about $1/60$ and $1/80$, because the factorials make its terms shrink faster. That ratio is the practical rule for how many terms you need.

導讀 泰勒多項式在做什麼

把一個彎曲的函數，在某一點附近用多項式來近似：常數項對準高度，一次項對準斜率，二次項對準彎曲程度。在收斂範圍內，階數越高、離展開點越近，近似得越好。金融裡最重要的是二次項，因為 $\ln(1+x)$ 往下彎，所以「對數報酬 $\approx$ 簡單報酬」永遠有一個 $-x^2/2$ 的修正，後面所有的 $-\sigma^2/2$ 都從這裡來。

KNOWLEDGE CHECK 1

In $T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$, which coefficient would you change to make the approximation bend upward instead of downward, and what property of f does it encode?

ANSWER

The quadratic coefficient $\frac{1}{2}f''(a)$. Its sign is the sign of the curvature: $f''(a) > 0$ (convex) bends up, $f''(a) < 0$ (concave) bends down. $\ln(1+x)$ is concave, so its quadratic term is negative.

3 FOUR EXPANSIONS THAT RUN FINANCE

- ☐ derive the expansions of e^x , $\ln(1+x)$ and $(1+x)^\alpha$, and state where each one converges

Three Maclaurin series, plus the geometric series they all lean on, generate most approximations in this book.

The exponential. Every derivative of e^x is e^x , so $f^{(k)}(0) = 1$ for all k and

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \quad \text{for all real } x. \quad (3.2)$$

The logarithm. Differentiate $f(x) = \ln(1+x)$ repeatedly: $f'(x) = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f'''(x) = 2(1+x)^{-3}$, and in general $f^{(k)}(x) = (-1)^{k+1}(k-1)!(1+x)^{-k}$. At 0 this gives $f^{(k)}(0)/k! = (-1)^{k+1}/k$, so

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{for } -1 < x \leq 1. \quad (3.3)$$

A second derivation explains the range. For $|t| < 1$ the geometric series gives $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots$. Integrating from 0 to x term by term yields (3.3) for $|x| < 1$. At $x = 1$ the alternating series still converges, to $\ln 2$ (by Abel's theorem), which is why the range is $-1 < x \leq 1$.

The binomial series. For any real exponent α , $f(x) = (1+x)^\alpha$ has $f^{(k)}(0) = \alpha(\alpha-1)\dots(\alpha-k+1)$, so

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots \quad \text{for } |x| < 1, \quad (3.4)$$

where $\binom{\alpha}{k} = \alpha(\alpha-1)\dots(\alpha-k+1)/k!$. With $\alpha = -1$ it is the alternating geometric series $1/(1+x) = 1 - x + x^2 - \dots$. With $\alpha = \frac{1}{2}$ it gives $\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$. When α is a positive integer the series stops and (3.4) is the ordinary binomial theorem.

Exhibit 2 collects the four series with their ranges and typical uses.

Exhibit 2: The expansions most used in finance

Function	Series about 0	Converges for	Typical use
e^x	$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$	all x	continuous compounding, $R = e^r - 1$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$	$-1 < x \leq 1$	log returns, growth rates, Kelly
$(1+x)^\alpha$	$1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots$	$ x < 1$	annualizing, real rates, leverage
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \dots$	$ x < 1$	annuities and perpetuities (LM6)

The radius of convergence has a financial meaning

The series (3.3) diverges for $x > 1$ and the function itself does not exist at $x = -1$. Both edges matter. At $x = -1$, a return of -100% , the log of wealth is $-\infty$: that is **ruin**. It is why log-utility investors such as Kelly bettors (LM41) never accept a position that can lose everything.

For $x > 1$, $\ln(1+x)$ is perfectly finite, but no number of series terms will reach it. A truncated series is a local tool; for large moves, compute the exact function.

PITFALL Series approximations for large returns

For a +150% year, $x - x^2/2 = 1.5 - 1.125 = 0.375$, while $\ln 2.5 = 0.916$. Adding terms makes it worse, because the series diverges for $x > 1$. Use $\ln(1 + R)$ itself whenever $|R|$ is not small. A rough working limit for the second-order formula is $|R| \leq 0.1$, where its error stays below 4 bp of log return (Section 4).

EXAMPLE 2

Real returns and the Fisher approximation

A one-year time savings deposit at Bank of Taiwan pays a fixed nominal rate $i = 1.725\%$, the rate posted in October 2026. Expected inflation is $\pi = 1.8\%$, an illustrative figure. Using the exact relation of LM1, here $1 + R^{\text{real}} = (1 + i)/(1 + \pi)$, compute the real rate exactly and with first- and second-order approximations.

SOLUTION

Exactly, $R^{\text{real}} = 1.01725/1.018 - 1 = -0.07367\%$.

Expanding $1/(1 + \pi) = 1 - \pi + \pi^2 - \dots$ gives $R^{\text{real}} = (i - \pi)(1 - \pi + \pi^2 - \dots)$. To first order $R^{\text{real}} \approx i - \pi = -0.07500\%$. Keeping the next term, $R^{\text{real}} \approx (i - \pi) - \pi(i - \pi) = -0.07365\%$, which matches the exact value to within 0.0001 percentage points. The first-order rule $i - \pi$ is the familiar Fisher approximation.

KNOWLEDGE CHECK 2

Using (3.4), write the second-order approximation of $(1 + x)^{1/12}$ and use it to convert an annual return of 9% into a monthly geometric return.

ANSWER

$(1 + x)^{1/12} \approx 1 + \frac{x}{12} + \frac{\frac{1}{12}(\frac{1}{12} - 1)}{2}x^2 = 1 + \frac{x}{12} - \frac{11}{288}x^2$. With $x = 0.09$: $1 + 0.0075 - 0.000309 = 1.007191$, so about 0.7191% per month; the exact value $1.09^{1/12} - 1 = 0.7207\%$. The neglected third-order term is positive, which is why the approximation is slightly low.

4

HOW GOOD IS “ $\approx$ ”? THE REMAINDER TERM

- ☐ bound the truncation error with the Lagrange remainder, and decide when a truncated expansion can be trusted

An approximation without an error estimate is a guess. Taylor's theorem supplies the estimate.

THEOREM Taylor's theorem with Lagrange remainder

Let f have $n + 1$ continuous derivatives on an interval containing a and x . Then

$$f(x) = T_n(x) + E_n(x), \quad E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} \quad (3.5)$$

for some ξ strictly between a and x .

Read (3.5) as: *the error is the next term of the series, evaluated at an unknown point ξ* . We rarely know ξ , but we usually know how large $f^{(n+1)}$ can be on the interval, and that bounds the error. In **big-O notation** we write $E_n(x) = O(|x-a|^{n+1})$: halve the distance to a and an order- n error shrinks by a factor of about 2^{n+1} .

The error of “log return $\approx R - R^2/2$ ”

For $f(x) = \ln(1+x)$, $f'''(\xi) = 2/(1+\xi)^3$, so with $n = 2$ and $a = 0$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\xi)^3}.$$

For $x > 0$, $\xi \in (0, x)$ gives $1 + \xi > 1$, and the error lies in $(0, x^3/3)$. For $-1 < x < 0$, $\xi \in (x, 0)$ gives $1 + \xi > 1 + x$, and the error is negative with size at most $|x|^3/(3(1+x)^3)$. Losses are approximated less accurately than gains of the same size, because $\ln(1+x)$ bends harder on the left. Exhibit 3 compares the actual errors with these bounds.

Exhibit 3: Second-order approximation of $\ln(1+x)$: actual error and Lagrange bound

x	$\ln(1+x)$	$T_2 = x - x^2/2$	Error $f - T_2$	Lagrange bound
-0.20	-0.223144	-0.220000	-0.003144	0.005208
-0.10	-0.105361	-0.105000	-0.000361	0.000457
-0.05	-0.051293	-0.051250	-0.000043	0.000049
-0.01	-0.010050	-0.010050	-0.0000003	0.0000003
+0.01	0.009950	0.009950	+0.0000003	0.0000003
+0.05	0.048790	0.048750	+0.000040	0.000042
+0.10	0.095310	0.095000	+0.000310	0.000333
+0.20	0.182322	0.180000	+0.002322	0.002667
+0.50	0.405465	0.375000	+0.030465	0.041667

Note: the bound is $|x|^3/3$ for $x > 0$ and $|x|^3/(3(1+x)^3)$ for $x < 0$. Every actual error lies inside its bound, and the bound is tight for small $|x|$.

EXAMPLE 3

The two worst days in 0050's history

On 5 August 2024, 0050 fell 9.130%. On 7 April 2025, after the US tariff announcement, it fell the full daily limit of 10.000%. For each day, compute the log return exactly and with T_1 and T_2 , and check the error against the Lagrange bound.

SOLUTION

5 August 2024, $R = -0.091299$: exact $\ln(1 + R) = -0.095740$. $T_1 = -0.091299$ is off by 44.4 bp; $T_2 = -0.095467$ is off by 2.7 bp; the bound is $0.091299^3 / (3 \times 0.908701^3) = 3.4$ bp.

7 April 2025, $R = -0.100000$: exact -0.105361 . T_1 is off by 53.6 bp; $T_2 = -0.105000$ is off by 3.6 bp, inside the bound of 4.6 bp.

Even on record crash days, the second-order formula is good to a few basis points of log return. The first-order formula is not: treating a -10% day as a -10% log return understates the damage by more than half a percentage point.

The title claims 0050's whole history, although the data start in October 2014. Before 1 June 2015 Taiwan's daily price limit was 7%, so no earlier day could fall this far. The next-worst day in the data is -7.03% (11 October 2018).

EXAMPLE 4

Why the “rule of 72” works

Money growing at rate R per year doubles in $\tau_2 = \ln 2 / \ln(1 + R)$ years. Expand the denominator to explain where 72 comes from and why it works best near 8%.

SOLUTION

From (3.3), $\ln(1 + R) = R(1 - \frac{R}{2} + \frac{R^2}{3} - \dots)$. Inverting the bracket with the geometric series of Exhibit 2 and keeping terms of first order in R :

$$\tau_2 \approx \frac{\ln 2}{R} \left(1 + \frac{R}{2}\right) = \frac{\ln 2}{R} + \frac{\ln 2}{2} \approx \frac{0.693}{R} + 0.347.$$

At $R = 6\%$ this gives 11.899 years with the unrounded constants (11.897 with 0.693 and 0.347), against the exact 11.896. In the form “numerator $\div 100R$ ” the numerator is $100 \ln 2 (1 + R/2) \approx 69.3 + 34.7R$, which equals 72 at $R = 2(72/(100 \ln 2) - 1) = 7.75\%$; the rule of 72 is exact at $R = 7.85\%$. It is a second-order Taylor approximation tuned for returns near 8%.

Step 1 (integral form). By the fundamental theorem of calculus, $f(x) = f(a) + \int_a^x f'(t) dt$. Integrate by parts, differentiating $f'(t)$ and taking $-(x-t)$ as the antiderivative of 1:

$$\int_a^x f'(t) dt = [-f'(t)(x-t)]_a^x + \int_a^x f''(t)(x-t) dt = f'(a)(x-a) + \int_a^x f''(t)(x-t) dt.$$

Repeating n times gives $f(x) = T_n(x) + E_n(x)$ with

$$E_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t) (x-t)^n dt.$$

Step 2 (Lagrange form). Suppose $x > a$ (the other case is symmetric). The weight $(x-t)^n$ is non-negative on $[a, x]$ and $f^{(n+1)}$ is continuous, so by the mean value theorem for integrals there is $\xi \in (a, x)$ with

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{n!} \int_a^x (x-t)^n dt = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}. \quad \blacksquare$$

The integral form is exact and contains no unknown point, which makes it convenient when a sharper bound is needed.

導讀 「 $\approx$ 」到底差多少

泰勒定理說：誤差 = 「下一項」，只是導數要在某個不知道的點 ξ 上取值。我們不知道 ξ 在哪，但能估出導數的上限，所以能給出誤差上界。以二階的 $\ln(1+x) \approx x - x^2/2$ 為例，連 2025/4/7 跌停（-10%）這種極端日，誤差也只有 3.6 bp；但若用一階的「對數報酬 $\approx$ 簡單報酬」，誤差會超過 50 bp。另外注意：同樣幅度，**下跌的誤差比上漲大**，因為 $\ln$ 在左邊彎得更厲害。

KNOWLEDGE CHECK 3

The fourth-order Taylor polynomial of e^x about 0 is used to compute $e^{-0.3}$. Bound the error.

ANSWER

$E_4(-0.3) = e^\xi(-0.3)^5/5!$ with $\xi \in (-0.3, 0)$, so $e^\xi < 1$ and $|E_4(-0.3)| < 0.3^5/120 = 2.025 \times 10^{-5}$. The actual error is $e^{-0.3} - T_4(-0.3) = 0.7408182 - 0.7408375 = -1.93 \times 10^{-5}$, inside the bound.

- relate simple and log returns, and arithmetic and geometric means, through second-order expansions, and quantify volatility drag

A holding-period (simple) return and a log return describe the same price move:

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1, \quad r_t = \ln(1 + R_t).$$

From (3.3) and (3.2), each is a series in the other:

$$r = R - \frac{R^2}{2} + \frac{R^3}{3} - \dots, \quad R = e^r - 1 = r + \frac{r^2}{2} + \frac{r^3}{6} + \dots \quad (3.6)$$

Because $\ln(1 + x)$ is concave, it lies below its tangent line at the origin, which is x itself. Hence $\ln(1 + x) \leq x$ for every $x > -1$, with equality only at 0 (Practice Problem 5 proves it with (3.5)). So the log return is always the smaller number, by about $R^2/2$ (Exhibit 4).

Exhibit 4: Simple versus log returns

Simple return R	Log return $r = \ln(1 + R)$	Gap $R - r$	Second-order gap $R^2/2$
-50%	-69.31%	19.31%	12.50%
-20%	-22.31%	2.31%	2.00%
-10%	-10.54%	0.54%	0.50%
-1%	-1.005%	0.005%	0.005%
+1%	+0.995%	0.005%	0.005%
+10%	+9.53%	0.47%	0.50%
+20%	+18.23%	1.77%	2.00%
+50%	+40.55%	9.45%	12.50%
+100%	+69.31%	30.69%	50.00%

Note: the second-order gap is accurate below about 10% and useless beyond 50%; the asymmetry between $\pm 50\%$ is mostly the third-order term $R^3/3$.

Two aggregation rules, two return types

Each definition is additive along a different dimension, which is why both are needed.

- **Across time, log returns add.** Over T periods the growth factor is $G = \prod_t (1 + R_t)$, so $\ln G = \sum_{t=1}^T r_t$, because the log of a product is a sum. Simple returns compound instead: $1 + R_{0 \rightarrow T} = G$.
- **Across assets, simple returns add.** A portfolio with weights w_j earns $R_p = \sum_j w_j R_j$ exactly. Log returns do not: $r_p = \ln(\sum_j w_j e^{r_j}) \neq \sum_j w_j r_j$.

The working rule follows. Use log returns for research along the time axis — trends, volatility, cross-asset comparisons. Use simple returns whenever money is aggregated across positions — portfolio accounting, settlement, drawdown in currency.

The gap in real data

The sample is the 2,902 days on which both 0050 and 00631L traded, from November 2014 to October 2026. Each fund was halted for a few sessions around its split, so two of these returns span several sessions.

Over these days, 0050's mean daily simple return was 0.0871% and its mean daily log return 0.0794%. The difference, 0.0077% (0.77 bp) a day, is what (3.6) predicts. Half the mean squared return is 0.771 bp, and half the variance is 0.767 bp. Summed over the sample's $A = 243.1$ returns a year, this “small” daily gap is roughly 1.9 percentage points of annual return.

PYTHON

```
import numpy as np

# ret: the lab's daily total returns (columns date, R_0050, R_00631L)
# on the 2,902 days when both funds traded; Part 2 of the lab builds it
# from the FinMind files in its data/ folder
R = ret["R_0050"].to_numpy()
r = np.log1p(R)                                # exact log returns
gap = (R.mean() - r.mean()) * 1e4              # in bp
half_m2 = (R**2).mean() / 2 * 1e4              # in bp
print(f"mean R      = {R.mean():.6%}")         # 0.087120%
print(f"mean r      = {r.mean():.6%}")         # 0.079422%
print(f"gap         = {gap:.3f} bp")            # 0.770 bp
print(f"mean(R^2)/2 = {half_m2:.3f} bp")        # 0.771 bp, eq. (3.6)
```

KNOWLEDGE CHECK 4

A portfolio holds 60% in asset A, which rose 10%, and 40% in asset B, which fell 10%. Compute the portfolio's simple return and log return, and the weighted average of the two log returns. Which aggregation is wrong?

ANSWER

$R_p = 0.6(0.10) + 0.4(-0.10) = 2.00\%$ exactly, so $r_p = \ln 1.02 = 1.980\%$. The weighted average of log returns is $0.6 \ln 1.1 + 0.4 \ln 0.9 = 0.6(0.09531) + 0.4(-0.10536) = 1.504\%$, which is wrong. Log returns do not aggregate across assets.

- relate simple and log returns, and arithmetic and geometric means, through second-order expansions, and quantify volatility drag

Given returns $R_1, \dots, R_T$, three averages answer three different questions.

$$\bar{R} = \frac{1}{T} \sum_{t=1}^T R_t, \quad R_G = \left(\prod_{t=1}^T (1 + R_t) \right)^{1/T} - 1, \quad g = \ln(1 + R_G) = \frac{1}{T} \sum_{t=1}^T \ln(1 + R_t).$$

$\bar{R}$ is the average one-period outcome. R_G is the constant return that produces the same ending wealth. g is the same thing on the log scale, where it adds up over time. Wealth compounds, so R_G and g , not $\bar{R}$, describe what a buy-and-hold investor actually earned.

Deriving the drag

Apply (3.3) to each period and average:

$$g = \frac{1}{T} \sum_t \left(R_t - \frac{R_t^2}{2} + \frac{R_t^3}{3} - \dots \right) = \bar{R} - \frac{1}{2} \cdot \frac{1}{T} \sum_t R_t^2 + \frac{1}{3} \cdot \frac{1}{T} \sum_t R_t^3 - \dots$$

The mean of squares needs no probability theory. With the population variance $s^2 = \frac{1}{T} \sum_t (R_t - \bar{R})^2$, expanding the square gives the identity

$$\frac{1}{T} \sum_{t=1}^T R_t^2 = s^2 + \bar{R}^2. \quad (3.7)$$

Substituting and stopping at second order:

$$g \approx \bar{R} - \frac{s^2 + \bar{R}^2}{2}. \quad (3.8)$$

Converting back with (3.2), $R_G = e^g - 1 \approx g + \frac{g^2}{2}$, and $g^2 \approx \bar{R}^2$ to this order, so the $\bar{R}^2$ terms cancel:

$$R_G \approx \bar{R} - \frac{s^2}{2}. \quad (3.9)$$

This is the familiar rule of thumb “geometric mean $\approx$ arithmetic mean $- \sigma^2/2$ ”. The gap $\bar{R} - R_G \approx s^2/2$ is called **volatility drag**. Equations (3.8) and (3.9) differ: (3.9) approximates the geometric *simple* return, while (3.8) approximates the *log* growth rate and keeps an extra $-\bar{R}^2/2$. Both drop third- and higher-order terms, so both are approximations.

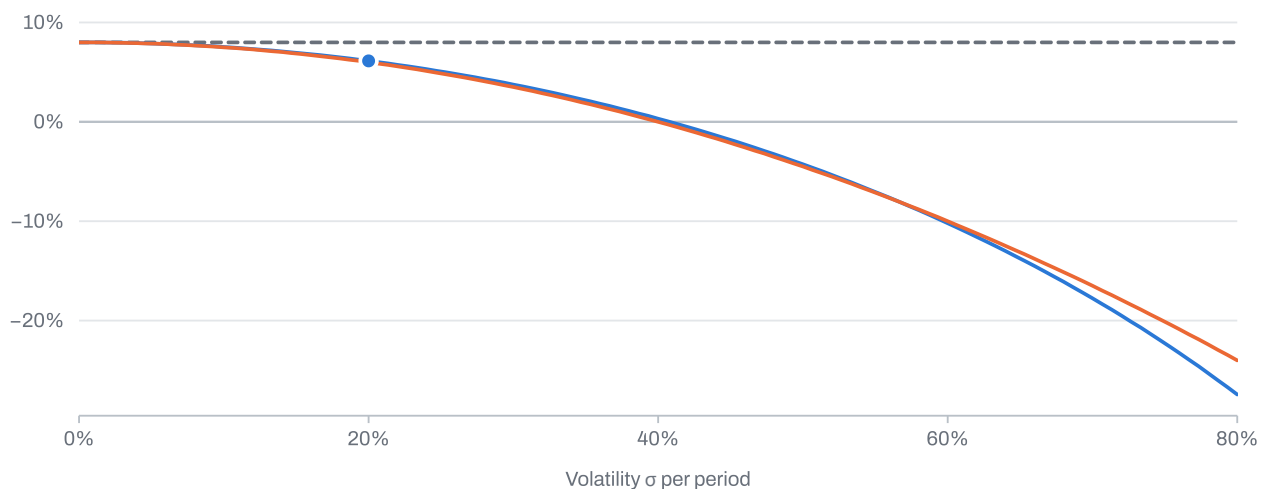
RESULT Volatility drag

To second order, the geometric mean return equals the arithmetic mean minus half the variance, $R_G \approx \bar{R} - s^2/2$, and the log growth rate is $g \approx \bar{R} - (s^2 + \bar{R}^2)/2$. The neglected terms are of third order in the returns. They grow with the length of the period and the size of the moves. They depend on the mean of the returns and on their **skewness**, the asymmetry of moves around the mean (LM19).

Volatility drag in a two-outcome world

INTERACTIVE

--- Arithmetic mean μ — Exact geometric return — Approximation $\mu - \sigma^2/2$



Exact geometric

6.132%

$\mu - \sigma^2/2$

6.000%

Approx. error

-13.2 bp

Log growth g

5.951%

Mean wealth after T periods

4.66x

Median wealth after T periods

3.29x

Each period the return is $\mu + \sigma$ or $\mu - \sigma$ with equal probability, so σ is exactly the standard deviation. The exact geometric return is $\sqrt{((1+\mu)^2 - \sigma^2)} - 1$.

How accuracy depends on the measurement period

The approximation error depends on how large the per-period returns are, so it depends on the sampling frequency. Exhibit 5 measures it on the same 0050 history at five frequencies.

Exhibit 5: Volatility-drag approximations on 0050, November 2014 – October 2026

Period	T	$\bar{R}$	s	R_G exact	$\bar{R} - \frac{s^2}{2}$	Error (bp)	g exact	$\frac{\bar{R} - \frac{s^2 + \bar{R}^2}{2}}{2}$	Error (bp)
Daily	2,902	0.087120%	1.2385%	0.0795%	0.0795%	−0.00	0.0794%	0.0794%	−0.00
Weekly	615	0.40652%	2.4953%	0.3755%	0.3754%	−0.01	0.3748%	0.3746%	−0.02
Monthly	144	1.7683%	5.6489%	1.6135%	1.6087%	−0.5	1.6006%	1.5931%	−0.7
Quarterly	49	5.3673%	10.953%	4.816%	4.767%	−4.9	4.704%	4.623%	−8.0
Annual	11	18.606%	20.173%	16.72%	16.57%	−14.9	15.46%	14.84%	−62.1

Note: total-return series with dividends reinvested, on the 2,902 days common to 0050 and 00631L (Section 5). Calendar periods: the first quarter and the last month and quarter are partial, and the annual row uses only the full years 2015–2025. T is the number of returns at each frequency and s the population standard deviation, as in (3.7). Each error is the approximation minus the exact value, in basis points, computed before rounding.

At daily frequency the approximation is exact to the displayed precision. At annual frequency, where single observations reach +48.7% and −21.3%, the error of (3.9) is 15 bp and that of (3.8) is 62 bp. The rule: second-order formulas are reliable when per-period standard deviations are a few percent, and need checking when they approach 20%.

EXAMPLE 5

Five years of 0050

0050's calendar-year total returns for 2020–2024 were 31.08%, 21.97%, −21.34%, 27.40% and 48.67%. Compute the arithmetic mean, the exact geometric mean and the approximation (3.9).

SOLUTION

$\bar{R} = (31.08 + 21.97 - 21.34 + 27.40 + 48.67)/5 = 21.56\%$. The population variance of the five returns is $s^2 = 0.05401$ ($s = 23.24\%$), so (3.9) gives $21.56\% - 2.70\% = 18.86\%$.

Exactly, $R_G = (1.3108 \times 1.2197 \times 0.7866 \times 1.2740 \times 1.4867)^{1/5} - 1 = 18.96\%$. The approximation is 10 bp low. An investor who quoted the 21.6% arithmetic mean as “what 0050 earned” would overstate the realized compound return by 2.6 percentage points a year.

EXAMPLE 6

The +2% / −1% bet, revisited

Take the bet of LM2, which returns +2% or −1% of the stake with equal probability. Stake the whole capital on every trade, reinvesting gains. Capital then rises 2% or falls 1% per trade. Compute the arithmetic mean, the exact geometric return, the log growth rate, and both approximations.

SOLUTION

$\bar{R} = 0.5\%$ and $s^2 = \frac{1}{2}(0.015^2 + 0.015^2) = 0.000225$. Exactly, $R_G = \sqrt{1.02 \times 0.99} - 1 = 0.48881\%$ and $g = \frac{1}{2}(\ln 1.02 + \ln 0.99) = 0.48761\%$.

Equation (3.9) gives $0.5\% - 0.01125\% = 0.48875\%$, off by 0.006 bp. Equation (3.8) gives $0.5\% - (0.000225 + 0.000025)/2 = 0.48750\%$, off by 0.011 bp. With small moves both are excellent, but the drag is real: about 2.2% of the arithmetic edge disappears to compounding.

PITFALL “Geometric = arithmetic – $\sigma^2/2$ ” is not an identity

Take returns of +100% and –50% with equal probability. The arithmetic mean is +25%, but $(2 \times 0.5)^{1/2} - 1 = 0\%$ exactly: wealth goes nowhere. The approximation gives $25\% - 0.75^2/2 = -3.1\%$. With moves this large the neglected third- and fourth-order terms dominate.

For position sizing, compute $\frac{1}{T} \sum \ln(1 + R_t)$ directly rather than plugging into $\mu - \sigma^2/2$ – that is exactly the empirical-Kelly procedure of LM45.

導讀 波動就是成本

同樣的算術平均報酬，波動越大，複利後實際賺到的（幾何平均）越少，差額約等於 $s^2/2$ ，這就是「波動拖累」。它來自 $\ln(1+x)$ 的二次項，所以是二階近似：日報酬時幾乎完全精準；年報酬（單年 +48.7%、–21.3% 這種大波動）時就會有 15 到 62 bp 的誤差。要決定部位大小時，請直接算 $\frac{1}{T} \sum \ln(1 + R_t)$ ，不要只套 $\mu - \sigma^2/2$ 。另外記得：(3.9) 近似的是「幾何平均簡單報酬」 R_G ，(3.8) 近似的是「對數成長率」 g ，兩者差一個 $\bar{R}^2/2$ 。

KNOWLEDGE CHECK 5

Two funds have the same arithmetic mean annual return of 10%. Fund A has annual volatility 10%, fund B 30%. Estimate each fund's geometric mean return with (3.9). Over 20 years, roughly how many times as much wealth as B does A produce?

ANSWER

A: $10\% - 0.1^2/2 = 9.5\%$. B: $10\% - 0.3^2/2 = 5.5\%$. Over 20 years, $(1.095/1.055)^{20} = 2.10$: about twice the wealth, from identical arithmetic means. At 30% volatility the estimate for B is rough, beyond the 20% guide above, but the conclusion survives: with (3.8) the ratio is $e^{20 \times 0.04} = 2.23$.

- decompose the long-run return of a daily-reset leveraged ETF into a power of the index, a variance drag and a residual, and estimate each term from data

The daily-reset identity

A daily-reset fund with leverage L promises L times the index's *daily* simple return. Ignore costs for now and write the fund's daily return as LR_t . Its log return minus L times the index's log return is, by (3.3),

$$\ln(1 + LR) - L \ln(1 + R) = -\frac{L^2 - L}{2} R^2 + \frac{L^3 - L}{3} R^3 - \frac{L^4 - L}{4} R^4 + \dots \quad (3.10)$$

The series converges when $|LR| < 1$ and $|R| < 1$, which daily moves satisfy by a wide margin. The first-order terms cancel exactly – that is what “ $L \times$ daily” buys you. What remains starts at second order.

Now sum over the T days of a holding period and keep terms to second order. With G the index's growth factor, G_{fund} the fund's and $\text{RV} = \sum_t R_t^2$ the index's **realized variance**, this gives

$$G_{\text{fund}} \approx G^L \exp\left(-\frac{L(L-1)}{2} \text{RV}\right). \quad (3.11)$$

RESULT What a daily-reset fund delivers

Before costs, an $L \times$ daily fund returns approximately the L -th **power** of the index's growth factor, multiplied by a **variance drag** of $\exp(-\frac{L(L-1)}{2} \text{RV})$. For $L = 2$ the drag factor is $e^{-\text{RV}}$; for $L = 3$ it is $e^{-3\text{RV}}$. The drag depends on the path only through realized variance, plus small third-order terms in $\sum_t R_t^3$. Those terms reflect both the trend and the skewness of daily moves.

This dissolves the paradox. “Twice the index's return” means $1 + 2(G - 1)$, but for $L > 1$ the power satisfies $G^L \geq 1 + L(G - 1)$, with equality only at $G = 1$. This is Bernoulli's inequality, itself a statement that x^L lies above its tangent line at $x = 1$. So the fund has two forces: compounding of the trend, G^L versus $1 + L(G - 1)$, which favours the fund, and variance drag, which hurts it.

Strong trends with low volatility favour the fund; choppy markets with high volatility punish it. Practice Problem 9 makes this precise. Before costs, a $2 \times$ fund beats twice the index's simple return roughly when the trend $|1 - 1/G|$ exceeds the realized volatility $\sqrt{\text{RV}}$.

Twelve years of 00631L

Exhibit 6 compares three quantities built from the same 2,902 days. They are 0050's total-return growth, an exact daily $2 \times$ of 0050, $\prod_t (1 + 2R_t)$, and the Taylor formula $G^2 e^{-\text{RV}}$ along the path. 00631L's market price is plotted alongside.

Exhibit 6: Growth of one unit, November 2014 – October 2026 (log scale)

00631L (market price), end 44.54 Exact daily 2× of 0050, end 64.223 Taylor: $G^2 \cdot e^{(-\Sigma R^2)}$, end 64.220
 0050 total return, end 10.02

Growth of 1 unit (log scale)



Note: the dashed Taylor curve lies on top of the exact daily-2× curve for the whole period. At the end the Taylor formula gives 64.220 and the exact daily 2× 64.223; the legend gives each curve's end value. The gap between them and 00631L is not explained by (3.11): it is cost and replication basis.

Exhibit 7: Decomposing 00631L's log growth, 2014-10-31 → 2026-10-08

Component	Log points	Growth factor
$2 \times \text{log growth of 0050 } (2 \times 2.30483)$	4.6097	$\times 100.45 (= 10.0225^2)$
Variance drag, exact daily rebalancing $\sum [\ln(1 + 2R_t) - 2 \ln(1 + R_t)]$	-0.44729	$\times 0.639$
– second-order Taylor term $-RV$	-0.44735	
– third-order Taylor $-RV + 2 \sum R_t^3$	-0.44461	
– fourth-order Taylor $-RV + 2 \sum R_t^3 - 3.5 \sum R_t^4$	-0.44728	
Residual: fees, financing, replication basis, price vs NAV	-0.3659	$\times 0.694$
00631L log growth	3.7965	$\times 44.54$

Note: the first five rows use only 0050 data. The residual is 00631L's actual log growth minus the exact daily-2× growth. Growth factors are rounded; the product of the unrounded factors is 44.54.

Three conclusions follow from Exhibit 7. First, over the full sample the second-order term $-RV$ comes within 0.0001 log points of the exact rebalancing effect. That is partly luck: the third-order term $2 \sum R_t^3 = +0.0027$ and the fourth-order term $-3.5 \sum R_t^4 = -0.0027$ cancel. The same cancellation explains why the third-order row alone lands farther away.

Single calendar years differ by up to 0.0025 log points (2026 to date), and the running gap reached 0.0046 on 9 April 2025. So (3.11) is accurate to a few thousandths of a log point, which against a drag of 0.447 still makes it an excellent model.

Second, the drag averaged $0.4473/11.94 = 0.0375$ log points a year, essentially 0050's annualized variance, $s^2 A = (1.2385\%)^2 \times 243.1 = 0.0373$; the small difference is $\bar{R}^2 A = 0.0002$. Third, 00631L still grew 44-fold against 10-fold for 0050, because $G^2 \approx 100$ is a very large factor: the trend term overwhelmed the drag.

The year-by-year view in Exhibit 8 shows the two forces trading places.

Exhibit 8: Year-by-year decomposition of 00631L against twice 0050

Year	0050 total return	00631L	2 × 0050	Taylor drag −RV	Exact rebalancing effect	Residual
2014 (part)	2.374%	6.58%	4.75%	−0.0032	−0.0032	+0.0200
2015	−6.309%	−16.54%	−12.62%	−0.0266	−0.0265	−0.0239
2016	19.644%	39.18%	39.29%	−0.0224	−0.0224	−0.0058
2017	18.134%	39.82%	36.27%	−0.0088	−0.0087	+0.0106
2018	−4.951%	−10.38%	−9.90%	−0.0270	−0.0280	+0.0200
2019	33.516%	70.87%	67.03%	−0.0152	−0.0150	−0.0273
2020	31.084%	68.07%	62.17%	−0.0528	−0.0527	+0.0306
2021	21.971%	62.22%	43.94%	−0.0311	−0.0308	+0.1174
2022	−21.342%	−36.44%	−42.68%	−0.0476	−0.0477	+0.0747
2023	27.398%	62.93%	54.80%	−0.0194	−0.0191	+0.0230
2024	48.671%	58.96%	97.34%	−0.0611	−0.0627	−0.2670
2025	36.853%	50.99%	73.71%	−0.0595	−0.0603	−0.1552
2026 (part)	78.709%	147.95%	157.42%	−0.0727	−0.0702	−0.1830

Note: the rows marked (part) are partial years, from the base close of 31 October 2014 and to 8 October 2026. With them the log-point columns (drag, rebalancing effect, residual) add up to Exhibit 7, up to rounding. The rebalancing effect differs most from −RV in 2026, 2024 and 2018, by up to 0.0025. The third-order term $2 \sum R_t^3$ is negative when large down days dominate (2018 and 2024) and positive when large up days dominate (2026).

- **2017**, a steady uptrend with the lowest variance of any full year, is the textbook case for the power term. An exact daily 2× of 0050 would have returned 38.3% against twice the index's 36.3%.
- **2022** is the mirror image in a falling market. An exact daily 2× of 0050 would have lost 41.0%, less than twice 0050's 21.3% loss (42.7%). Compounding a downtrend also helps, because each day's 2×

loss is taken on a smaller base. 00631L itself lost only 36.4%, helped by a positive residual.

- **2015**, a choppy, slightly negative year, is where drag wins. An exact daily $2\times$ would have lost 14.5% and 00631L lost 16.5%, against twice the index's -12.6% .

What Taylor cannot see: the residual

Over the whole sample the residual averaged $-0.3659/11.94 = -0.031$ log points a year, about -3.0% of value. Exhibit 8 shows it swinging from $+0.117$ (2021) to -0.267 (2024), and January to 8 October 2026 alone contributed -0.183 . Over the eleven full years 2015–2025 the average was -0.018 . It is not a constant fee.

The residual contains management and trading costs, the financing embedded in the leverage, any gap between market price and NAV, and **replication basis**. 00631L builds most of its exposure with TAIEX futures rather than Taiwan 50 instruments. On 19 May 2026 it held TAIEX futures equal to 160.87% of net assets plus TSMC shares equal to 39.63%. TSMC's weight was then 43.7% in TAIEX and 60.9% in the Taiwan 50 index.

When TSMC leads, as in 2024–2026, a futures-heavy fund lags twice the Taiwan 50. In years when the broader market does relatively better, the residual can turn positive. A model explains only the terms it contains. Equation (3.11) prices the variance drag to within a few thousandths of a log point; the residual needs its own model.

EXAMPLE 7

TQQQ in 2022

In 2022 QQQ's adjusted close fell from 386.63 to 260.68 (-32.58%) and TQQQ's from 39.55 to 8.27 (-79.09%). QQQ's daily returns had realized variance $RV = 0.1030$, a realized volatility $\sqrt{RV}$ of 32.1%. Decompose TQQQ's log return with (3.10) for $L = 3$, measuring the residual against both the second-order drag and the exact effect. Use the lab's daily QQQ returns for the third-order term and the exact rebalancing effect.

SOLUTION

Index power: $G = 260.68/386.63$ and $3 \ln G = -1.18252$, so $G^3 = 0.3065$.

Variance drag: the second-order term is $-\frac{3}{2}RV = -3 \times 0.1030 = -0.3090$; adding the third-order term $8 \sum R_t^3$ gives -0.3107 . The exact rebalancing effect $\sum [\ln(1 + 3R_t) - 3 \ln(1 + R_t)]$ is -0.31325 .

Residual: TQQQ's actual log return was $\ln(8.27/39.55) = -1.56493$. Against the second-order drag the residual is $-1.56493 + 1.18252 + 0.30900 = -0.0734$; against the exact rebalancing effect it is $-1.56493 + 1.18252 + 0.31325 = -0.0692$, about 6.7% of value.

Roughly three-fifths of it is cost, about $0.0086 + 2 \times 0.0168 = 0.042$. The first term is the expense ratio of 0.86% after fee waivers (summary prospectus of 1 October 2022). The second is the cost of financing two extra units of exposure at 2022's average effective federal funds rate of 1.68%. Policy rates rose steeply during the year: the Federal Reserve lifted its target range from 0–0.25% to 4.25–4.50%.

Most of the rest, about $3 \times 0.0074 = 0.022$, is a benchmark difference. QQQ's adjusted close includes its dividends, worth 0.0074 log points in 2022. TQQQ, however, targets three times the daily return of the Nasdaq-100 Index (NDX), a price index without dividends. Against QQQ's unadjusted close the residual is -0.047 , about 0.005 beyond fees and financing: swap spreads, trading costs and other tracking differences.

The paradox resolves cleanly: “ $3 \times -32.6\% = -97.7\%$ ” compares the fund to the wrong benchmark. The right benchmark is $G^3 e^{-3RV} = 0.3065 \times 0.734 = 0.225$, a 77.5% loss before costs (77.6% with the exact rebalancing effect). On QQQ's unadjusted closes (397.85 to 266.28), which approximate the index TQQQ targets, it is $0.2998 \times 0.734 = 0.220$, a 78.0% loss.

PITFALL Two true statements about leveraged ETFs

“Leveraged ETFs decay” is true *relative to* G^L : for a given index growth G , more realized variance means less money. In fact, by Bernoulli's inequality $1 + LR_t \leq (1 + R_t)^L$ every day, so before costs a daily $L \times$ fund can never beat G^L . “Leveraged ETFs can beat $L \times$ ” is true *relative to* $1 + L(G - 1)$, because of the convexity of G^L .

Arguments about leveraged funds usually go wrong because the two sides use different benchmarks. Neither statement says whether holding one is wise; that is a sizing question, answered by the Kelly analysis of [LM42](#) and [LM46](#).

The variance term in (3.11), $-\frac{L(L-1)}{2}\sigma^2$ per unit time, will reappear in [LM42](#) as the curvature of the Kelly growth curve. A position held at a constant multiple of equity and rebalanced continuously gives up $\frac{1}{2}\sigma^2$ times the square of that multiple in log growth. A fund's leverage and a Kelly investor's exposure are the same variable seen from two sides.

導讀 正二到底「耗損」多少

每日重設的 L 倍基金，長期成長倍數 $\approx$ 指數成長倍數的 L 次方，再乘上變異數拖累 $e^{-\frac{L(L-1)}{2}RV}$ 。所以「正二會耗損」和「正二打敗兩倍報酬」都對，只是比較的基準不同：前者跟 G^2 比，後者跟 $1 + 2(G - 1)$ 比。以 0050 這 12 年的日報酬計算，二階泰勒項 -0.44735 與精確的每日再平衡效果 -0.44729 幾乎一樣（部分要歸功於三階與四階項剛好互相抵銷），平均每年拖累約 0.0375 個對數點，接近 0050 的年化變異數 0.0373。00631L 剩下的殘差平均每年約 -0.031 個對數點（約 -3.0% ），不是泰勒能解釋的，來自費用、槓桿融資、市價與淨值的差距，以及「用台指期複製台灣50」造成的基差。台積電在台灣50權重 60.9%、在加權指數只有 43.7%（2026/5/19），所以台積電領漲的年份（2024–2026）正二就會落後兩倍台灣 50。

KNOWLEDGE CHECK 6

An index ends a year exactly where it started ($G = 1$) after a year with realized variance $RV = 0.04$. Before costs, what does a $2\times$ daily fund return? A $3\times$ fund? What does “ L times the index's return” predict?

ANSWER

By (3.11), to second order the $2\times$ fund returns $\approx G^2 e^{-RV} - 1 = e^{-0.04} - 1 = -3.9\%$, and the $3\times$ fund $\approx G^3 e^{-3RV} - 1 = e^{-0.12} - 1 = -11.3\%$. “ L times 0%” predicts 0% for both. In a flat, volatile market, drag is all that remains.

8 THE SAME SECOND-ORDER IDEA ELSEWHERE

- ☐ apply first- and second-order approximations to bond prices (duration and convexity), and recognise the same structure in option delta-gamma and in Itô's lemma

Bond prices: duration and convexity

A bond's price $P(y)$ is a smooth, convex function of its yield. A second-order expansion about the current yield gives the standard approximation

$$\frac{\Delta P}{P} \approx -D_{\text{mod}} \Delta y + \frac{1}{2} C (\Delta y)^2, \quad D_{\text{mod}} := -\frac{1}{P} \frac{dP}{dy}, \quad C := \frac{1}{P} \frac{d^2 P}{dy^2}. \quad (3.12)$$

Modified duration is the slope term and convexity the curvature term: the same level-slope-curvature structure as Section 2.

EXAMPLE 8**A 10-year bond under a 100 bp shock**

A 10-year bond pays a 2% annual coupon and trades at par (yield 2%). It is the bond of LM2, priced per 100 of face value by $P(y) = \sum_{t=1}^{10} 2(1+y)^{-t} + 100(1+y)^{-10}$. Its modified duration is 8.9826 and its convexity 93.995. For yield changes of +100, -100 and +300 bp, compute the percentage price change exactly, by duration alone and by duration plus convexity.

SOLUTION

Δy	Exact	Duration only	Duration + convexity
+100 bp	-8.5302%	-8.9826%	-8.5126%
-100 bp	+9.4713%	+8.9826%	+9.4526%
+300 bp	-23.1652%	-26.9478%	-22.7180%

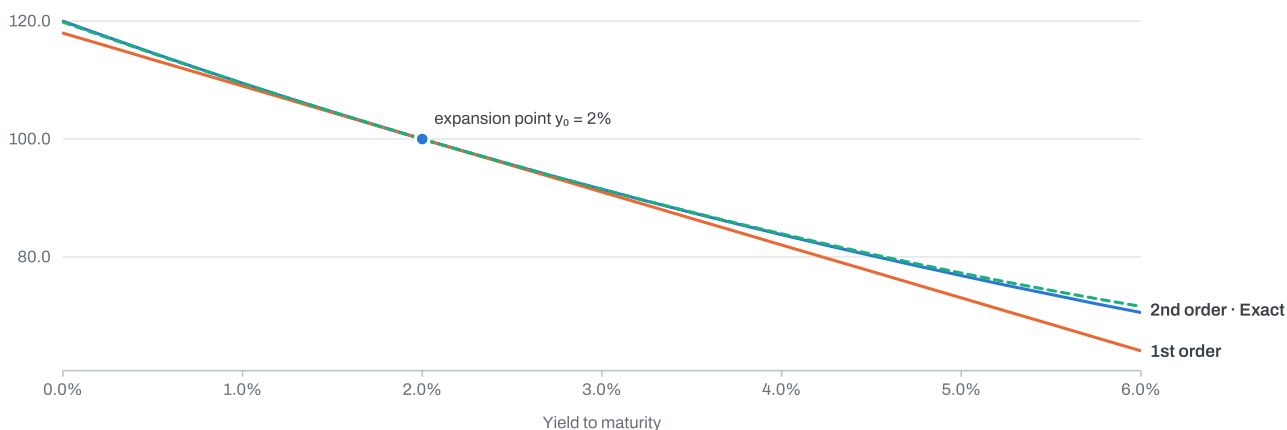
Duration alone is symmetric and misses the convexity gain on both sides. The second-order term cuts the error at ± 100 bp from 0.45–0.49 points to about 0.02. At +300 bp the second-order error grows to 0.4472 points, against 0.0176 at +100 bp. Three times the shock gives about 25 times the error, close to the factor of 27 that an $O(\Delta y^3)$ remainder implies.

Exhibit 9 shows the same comparison across all yields from 0% to 6%: the tangent line misses the curvature that the second-order term restores.

Exhibit 9: Price–yield curve with its first- and second-order approximations

— Exact price — Duration only (1st order) - - - Duration + convexity (2nd order)

Price per 100 face (10-year, 2% annual coupon)



Note: 10-year bond, 2% annual coupon, expanded about $y_0 = 2\%$ with $D_{\text{mod}} = 8.9826$ and $C = 93.995$.

Options: delta and gamma

An option's value $V(S, t)$ expanded to second order in the underlying price and first order in time gives the trader's daily P&L approximation

$$\Delta V \approx \Delta \cdot \Delta S + \frac{1}{2} \Gamma (\Delta S)^2 + \Theta \Delta t,$$

where $\Delta = \partial V / \partial S$, $\Gamma = \partial^2 V / \partial S^2$ and $\Theta = \partial V / \partial t$. A delta-hedged book removes the first term and keeps the gamma term: it earns $\frac{1}{2} \Gamma (\Delta S)^2$ from realized moves and pays Θ for the privilege. [LM31](#) develops this for TXO and warrants.

A preview of Itô's lemma

Apply a second-order expansion to $\ln S$ when the price follows a random walk with per-period return $\mu \Delta t + \sigma \sqrt{\Delta t} Z$, with Z a standard random shock:

$$\Delta \ln S \approx \frac{\Delta S}{S} - \frac{1}{2} \left(\frac{\Delta S}{S} \right)^2.$$

In ordinary calculus the squared term is negligible, because $(\Delta x)^2$ is much smaller than Δx . Here it is not: $(\Delta S/S)^2 \approx \sigma^2 Z^2 \Delta t$ is of the same order as the drift $\mu \Delta t$, and averages to $\sigma^2 \Delta t$. As $\Delta t \rightarrow 0$ the shocks $\sqrt{\Delta t} Z$ become the increments dB of a standard **Brownian motion** B_t ([LM27](#)), and keeping the squared term yields

$$d \ln S = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dB. \quad (3.13)$$

This is Itô's lemma for the logarithm, and the $-\sigma^2/2$ in it is the volatility drag of Section 6. In discrete time the drag was an approximation; in continuous time it becomes exact, because every higher-order term vanishes as $\Delta t \rightarrow 0$. [LM28](#) makes this rigorous, and [LM29](#) uses it to solve geometric Brownian motion.

DEEP DIVE Why $(\Delta B)^2$ behaves like Δt optional · click to expand

Here τ is a horizon in years and N a number of steps. Let $\Delta B = \sqrt{\Delta t} Z$ with Z standard normal. Then $\mathbb{E}[(\Delta B)^2] = \Delta t$ and $\text{Var}[(\Delta B)^2] = \Delta t^2 \text{Var}(Z^2) = 2\Delta t^2$. Split $[0, \tau]$ into $N = \tau/\Delta t$ steps and add the squared increments: the sum has mean τ and variance $N \cdot 2\Delta t^2 = 2\tau\Delta t$, which goes to 0 as $\Delta t \rightarrow 0$.

The sum of squared increments therefore converges to the deterministic number τ : the **quadratic variation** of Brownian motion. A smooth path has quadratic variation 0, because its squared increments are of order Δt^2 . That single difference is why stochastic calculus keeps the second-order Taylor term and ordinary calculus discards it.

KNOWLEDGE CHECK 7

Why does a long-only bond portfolio with positive convexity gain from a large yield move in *either* direction, relative to the duration-only estimate?

ANSWER

The convexity term $\frac{1}{2}C(\Delta y)^2$ is positive whatever the sign of Δy . Rates up: the loss is smaller than duration predicts. Rates down: the gain is larger. It is the bond-market analogue of a long-gamma option position.

9 TAYLOR APPROXIMATIONS IN PYTHON

☐ implement and verify Taylor approximations in Python

Two habits make approximations safe in code. First, compute the exact quantity alongside the approximation and report the gap. Second, prefer numerically stable library functions: `np.log1p(x)` and `np.expm1(x)` are accurate for tiny `x`, where `np.log(1 + x)` loses digits to rounding.

PYTHON

```
import math

def taylor_log1p(x, n):
    """Order-n Maclaurin polynomial of ln(1+x), eq. (3.3)."""
    return sum((-1) ** (k + 1) * x**k / k for k in range(1, n + 1))

def lagrange_bound_log1p(x, n):
    """Upper bound on |ln(1+x) - T_n(x)| from eq. (3.5)."""
    base = 1.0 if x >= 0 else 1.0 + x
    return abs(x) ** (n + 1) / ((n + 1) * base ** (n + 1))

for x in (-0.10, -0.0913, 0.05, 0.50):
    err = math.log1p(x) - taylor_log1p(x, 2)
    bound = lagrange_bound_log1p(x, 2)
    print(f"x={x:+.4f}  error={err:+.2e}  bound={bound:.2e}")
```

The decomposition of Section 7 takes a few vectorized lines for any index-fund pair with aligned daily simple returns:

PYTHON

```
import numpy as np

def leverage_decomposition(R, RF, L):
    """Split a daily-reset fund's log growth into power, drag and residual
    (eqs. 3.10-3.11). R, RF: aligned daily simple returns of the index and
    the fund, as NumPy arrays."""
    power = L * np.log1p(R).sum() #  $\ln G^L$ 
    exact_drag = (np.log1p(L * R) - L * np.log1p(R)).sum() # exact effect
    taylor2 = -L * (L - 1) / 2 * (R**2).sum() # to 2nd order
    taylor3 = taylor2 + (L**3 - L) / 3 * (R**3).sum() # to 3rd order
    taylor4 = taylor3 - (L**4 - L) / 4 * (R**4).sum() # to 4th order
    actual = np.log1p(RF).sum()
    return {"power": power, "exact_drag": exact_drag,
            "taylor2": taylor2, "taylor3": taylor3, "taylor4": taylor4,
            "residual": actual - power - exact_drag, "actual": actual}

# ret: the lab's aligned daily returns, built in its Part 2
# from the files in its data/ folder
R, RF = ret["R_0050"].to_numpy(), ret["R_00631L"].to_numpy()
d = leverage_decomposition(R, RF, L=2)
print({name: round(float(v), 5) for name, v in d.items()})
# power 4.60966, exact_drag -0.44729, taylor2 -0.44735, taylor3 -0.44461,
# taylor4 -0.44728, residual -0.36588, actual 3.79649
```

IN PYTHON Companion lab

The Polars-first notebook `LM03_lab.ipynb` rebuilds every exhibit and computational example in this module, and the numbers in its knowledge checks. The data-driven ones use FinMind price files: Exhibit 1, Exhibit 5, Exhibit 6 to Exhibit 8, Example 3, Example 5 and Example 7. Along the way it handles the 0050 split and dividends and the 00631L split. It also solves the computational practice problems and ends with an extension exercise on an inverse fund.

The notebook reads FinMind files that you download once with your own token. From the repository root, run `uv run python data/fetch_finmind.py`, then `uv run python data/make_lab_data.py`. FinMind's licence does not allow the book to redistribute the data. After that, the notebook runs offline.

SUMMARY

- A Taylor polynomial (3.1) matches a function's value and first n derivatives at one point. Its coefficients are $f^{(k)}(a)/k!$ because differentiating $(x - a)^k$ k times produces $k!$.

- The expansions of e^x , $\ln(1+x)$ and $(1+x)^\alpha$ generate most financial approximations. The series for $\ln(1+x)$ converges only on $(-1, 1]$; for large moves, use the exact function.
- The Lagrange remainder (3.5) makes every “ $\approx$ ” checkable: the error is the next term at an unknown point. For $\ln(1+x) \approx x - x^2/2$ the error stays below 4 bp of log return for daily moves up to 10%, including the 2024 and 2025 crash days.
- Log returns add across time; simple returns add across assets. The log return is smaller than the simple return by about $R^2/2$.
- Volatility drag: $R_G \approx \bar{R} - s^2/2$ and $g \approx \bar{R} - (s^2 + \bar{R}^2)/2$. Both are second-order approximations whose error grows with the size of per-period moves.
- A daily-reset $L \times$ fund earns approximately $G^L e^{-L(L-1)RV/2}$ before costs. On 0050 over the full 2014–2026 sample, the Taylor term (−0.44735) came within 0.0001 of the exact rebalancing effect (−0.44729). Cancelling third- and fourth-order terms helped; single calendar years differ by up to 0.0025. 00631L's further residual of −0.031 log points (about −3.0%) a year reflects costs and futures-based replication.
- Duration–convexity, delta–gamma and Itô's lemma share the level–slope–curvature structure. In Itô's lemma the squared term survives because $(\Delta B)^2$ is of order Δt .

PRACTICE PROBLEMS

1. The second-order Maclaurin polynomial of $f(x) = e^{2x}$ is:
 - A. $1 + 2x + x^2$
 - B. $1 + 2x + 2x^2$
 - C. $1 + x + \frac{x^2}{2}$
2. For a simple return of +8%, the second-order approximation $\ln(1+x) \approx x - x^2/2$ gives a log return *closest* to:
 - A. 7.68%
 - B. 7.70%
 - C. 8.00%
3. Approximate $\ln(0.92)$ with the second-order Taylor polynomial about 0. Compute the Lagrange bound on the error and the actual error.
4. A fund's annual returns over three years are +30%, −20% and +10%. Its geometric mean return is *closest* to:
 - A. 3.5%
 - B. 4.6%
 - C. 6.7%

5. Use the Lagrange form of the first-order remainder to prove that $\ln(1+x) \leq x$ for every $x > -1$, with equality only at $x = 0$.
6. A strategy returns +60% or -40% per period with equal probability. Compute the arithmetic mean, the exact geometric mean and the approximation $\bar{R} - s^2/2$. Explain the size of the approximation error.
7. An index rises 10% on day 1 and falls 10% on day 2. Compute the two-day return of the index and of a 2× daily fund (before costs). Then compute G^2 and the approximation (3.11), and comment on the result.
8. Over a year an index's growth factor is $G = 1.20$ and its realized variance is $RV = 0.09$. Before costs, a 3× daily fund's annual return is *closest* to:
 - A. +32%
 - B. +60%
 - C. +73%
9. Show that, before costs and to first order in RV , a 2× daily fund beats twice the index's simple return exactly when $|1 - 1/G| > \sqrt{RV}$. Check the condition for 0050 in 2017 ($G = 1.1813$, $RV = 0.0088$) and 2015 ($G = 0.9369$, $RV = 0.0266$).
10. A bond has modified duration 7.5 and convexity 70. Estimate its percentage price change if its yield rises by 150 bp.
11. Which statement about the Maclaurin series of $\ln(1+x)$ is correct?
 - A. It converges for every real x .
 - B. It converges for $-1 < x < 1$ only.
 - C. It converges for $-1 < x \leq 1$ only.
12. Derive the doubling-time approximation $\tau_2 \approx 0.693/R + 0.347$ and use it for $R = 6\%$. Compare with the exact value and the rule of 72.
13. Nominal return is 3% and inflation is 2.5%. Compute the real return exactly, with the first-order Fisher approximation, and with the second-order correction.
14. (Python) Apply `leverage_decomposition` from Section 9 to the lab's aligned return frame `ret`, restricted to the years 2020–2023 combined, to decompose 00631L's log growth. Which term changes sign compared with the full-period result?

SOLUTIONS

1. **B** is correct. $f(0) = 1$, $f'(0) = 2$ and $f''(0) = 4$, so $T_2 = 1 + 2x + \frac{4}{2}x^2 = 1 + 2x + 2x^2$. Equivalently, substitute $2x$ into (3.2).
2. **A** is correct. $0.08 - 0.08^2/2 = 0.08 - 0.0032 = 0.0768$. **B** is the exact $\ln 1.08 = 7.696\%$, which the question does not ask for; **C** is the first-order value x . The exact value exceeds the approximation by 0.016 points, close to the third-order term $x^3/3 = 0.017$ points.
3. $T_2(-0.08) = -0.08 - 0.0032 = -0.0832$; exactly, $\ln 0.92 = -0.0833816$. The error is -0.0001816 . For $x < 0$ the bound is $|x|^3/(3(1+x)^3) = 0.000512/(3 \times 0.778688) = 0.000219$, so the actual error is about 83% of the bound.
4. **B** is correct. $(1.30 \times 0.80 \times 1.10)^{1/3} - 1 = 1.144^{1/3} - 1 = 4.586\%$; as a check, with $\bar{R} = 6.667\%$ and population variance $s^2 = 0.04222$, (3.9) gives $6.667\% - 2.111\% = 4.556\%$. **C** is the arithmetic mean. **A** uses the sample variance, which divides by $T - 1 = 2$ instead of T as (3.7) does: $6.667\% - 0.06333/2 = 3.50\%$.
5. At $x = 0$ both sides are 0; for $x \neq 0$, $f(t) = \ln(1 + t)$ has a continuous second derivative on $(-1, \infty)$, which contains 0 and x . So (3.5) with $n = 1$ and $a = 0$ gives $\ln(1 + x) = x - \frac{x^2}{2(1+\xi)^2}$ for some ξ strictly between 0 and x . Since $1 + \xi > 0$ and $x^2 > 0$, the remainder is strictly negative and $\ln(1 + x) < x$. Hence $\ln(1 + x) \leq x$ for every $x > -1$, with equality only at $x = 0$.
6. $\bar{R} = 10\%$, $s = 50\%$, so $\bar{R} - s^2/2 = 10\% - 12.5\% = -2.5\%$. Exactly, $\sqrt{1.6 \times 0.6} - 1 = \sqrt{0.96} - 1 = -2.020\%$. The error of 0.48 percentage points is large because the two outcomes, $+60\%$ and -40% , sit $\pm 50\%$ around the mean. At that distance, third- and fourth-order terms are not negligible (compare Exhibit 3 and Exhibit 4).
7. Index: $1.10 \times 0.90 - 1 = -1.00\%$; the fund, $1.20 \times 0.80 - 1 = -4.00\%$, lost four times as much. $G^2 = 0.99^2 = 0.9801$, and $RV = 0.1^2 + 0.1^2 = 0.02$, so (3.11) gives $0.9801 e^{-0.02} = 0.9607$, a -3.93% return against the exact -4.00% . Of the fund's -0.0408 log return, -0.0201 is the power term $2 \ln 0.99$ and -0.0207 is drag. The second-order term $-RV = -0.0200$ captures the drag almost entirely, even with 10% daily moves.
8. **A** is correct. $G^3 e^{-3RV} = 1.728 \times e^{-0.27} = 1.728 \times 0.7634 = 1.319$, about $+32\%$. **B** is three times the index's simple return; **C** is $G^3 - 1$ without drag.
9. Before costs the fund's growth factor is $G^2 e^{-RV} \approx G^2(1 - RV)$; twice the index's simple return gives $1 + 2(G - 1) = 2G - 1$. The fund wins when $G^2 - G^2 RV > 2G - 1$, i.e. $(G - 1)^2 > G^2 RV$, i.e. $|1 - 1/G| > \sqrt{RV}$. For 2017: $|1 - 1/1.1813| = 0.153 > \sqrt{0.0088} = 0.094$, so the fund should win; indeed exact daily $2\times$ returned 38.3% against 36.3% . For 2015: $|1 - 1/0.9369| = 0.067 < \sqrt{0.0266} = 0.163$, so the fund should lose; exact daily $2\times$ returned -14.5% against -12.6% .
10. $-7.5 \times 0.015 + \frac{1}{2} \times 70 \times 0.015^2 = -0.1125 + 0.0079 = -10.46\%$.
11. **C** is correct (Section 3). At $x = 1$ the series converges to $\ln 2$, which **B** misses; at $x = -1$ the function is undefined and the series diverges. For $|x| > 1$ the terms grow without bound, which rules out **A**.

12. See Example 4. At $R = 6\%$: $\ln 2/0.06 + \ln 2/2 = 11.899$ years (11.897 with the rounded constants 0.693 and 0.347); exact $\ln 2 / \ln 1.06 = 11.896$; the rule of 72 gives 12.
13. Exact: $1.03/1.025 - 1 = 0.4878\%$. First order: $3\% - 2.5\% = 0.5\%$. Second order: $(i - \pi) - \pi(i - \pi) = 0.5\% - 0.0125\% = 0.4875\%$.
14. Over 2020–2023 the power term is $+0.94274$ and the exact rebalancing effect -0.15035 (second-order $-RV = -0.15086$, third-order -0.14979). The residual is $+0.24560$, so the actual log growth is $+1.03799$ (00631L +182.4%). The residual equals the sum of the four yearly residuals in Exhibit 8 (0.2457 at that table's rounding).

Against the full period (residual -0.3659) only the residual changes sign. The drag stays negative, as it must; the residual is driven largely by replication basis, which can help as well as hurt.

GLOSSARY

Term	中文	Meaning
Taylor polynomial	泰勒多項式	Polynomial matching a function's value and first n derivatives at a point, (3.1).
Maclaurin series	馬克勞林級數	Taylor series about 0.
Lagrange remainder	拉格朗日餘項	Error term $f^{(n+1)}(\xi)(x-a)^{n+1}/(n+1)!$ of an order- n approximation, (3.5).
Radius of convergence	收斂半徑	Distance from the expansion point within which a power series converges.
Big-O notation	大O符號	$O(x-a ^{n+1})$: a quantity bounded by a constant times $ x-a ^{n+1}$ as $x \rightarrow a$.
Simple return	簡單報酬	$R = (P_t + D_t)/P_{t-1} - 1$; adds across assets.
Log return	對數報酬	$r = \ln(1 + R)$, the continuously compounded return; adds across time.
Geometric mean return	幾何平均報酬	Constant per-period return producing the same terminal wealth.
Volatility drag	波動拖累	Arithmetic minus geometric mean, $\approx s^2/2$ to second order.
Realized variance	實現變異數	$RV = \sum_t R_t^2$ over a holding period.
Daily-reset leveraged ETF	每日重設槓桿型 ETF	Fund targeting L times the index's daily return.
Replication basis	複製基差	Return gap from tracking an index with different instruments, such as TAIEX futures for Taiwan 50.
Modified duration	修正存續期間	$-P'(y)/P$: first-order price sensitivity to yield.
Convexity	凸性	$P''(y)/P$: second-order price sensitivity to yield.
Delta, gamma	Delta、Gamma	First and second derivatives of an option's value with respect to the underlying price.
Quadratic variation	二次變分	Limit of the sum of squared increments of a path; τ for Brownian motion on $[0, \tau]$.

REFERENCES

- Stewart, J., D. K. Clegg, and S. Watson (2021). *Calculus: Early Transcendentals*, 9th ed. Cengage (ISBN 9781337613927). Chapter on infinite sequences and series (Taylor and Maclaurin series).

- Cheng, M., and A. Madhavan (2009). “The Dynamics of Leveraged and Inverse Exchange-Traded Funds.” *Journal of Investment Management* 7 (4).
- Avellaneda, M., and S. Zhang (2010). “Path-Dependence of Leveraged ETF Returns.” *SIAM Journal on Financial Mathematics* 1: 586–603.
- 姚惠茹 (26 May 2026). “00631L 曝台積電權重落差！決議增持權值股現貨「貼近指數兩倍報酬」.” TechNews 科技新報, 財經, 26 May 2026.
- 三立新聞網 (27 May 2026). “狂飆3311%！這檔正二增持「台積電」現貨 投信操作戰術曝光.” SETN, news 1844764.
- Taiwan Stock Exchange. Company history: the daily price fluctuation limit was raised from 7% to 10% on 1 June 2015. TWSE, company history, accessed 11 October 2026.
- Bank of Taiwan. New Taiwan dollar deposit rates: one-year time savings deposit, fixed rate 1.725%, in force since 1 August 2024. Bank of Taiwan, deposit rates, accessed 11 October 2026.
- ProShares Trust. Summary prospectus of ProShares UltraPro QQQ (TQQQ), 1 October 2022: objective of three times the daily return of the Nasdaq-100 Index (Bloomberg ticker NDX); total annual operating expenses 0.98% before and 0.86% after fee waivers. SEC EDGAR, summary prospectus.
- Nasdaq, Inc. NDX index versions: the Nasdaq-100 (NDX) is the price-return version, which leaves cash dividends out; the total-return version, XNDX, reinvests them on the ex-date. Nasdaq, NDX index versions (PDF), accessed 11 October 2026.
- Board of Governors of the Federal Reserve System. Open market operations: changes in the federal funds target range, 2022. Federal Reserve, open market operations, accessed 11 October 2026.
- Federal Reserve Bank of St. Louis. Effective federal funds rate, monthly, series FEDFUNDS; the twelve monthly values of 2022 average 1.68%. FRED, FEDFUNDS, accessed 11 October 2026.
- FinMind open data: `TaiwanStockPrice`, `TaiwanStockDividendResult` (0050, 00631L) and `USStockPrice` (QQQ, TQQQ), retrieved 10 October 2026.