

6

Sequences, Series and the Time Value of Money

One geometric series prices deposits, loans, bonds and stocks; no-arbitrage adds futures and turns the summer TX discount into an implied dividend yield.

VOLUME I · Calculus for Finance

BUILDS ON LM1 · LM3 · LM4

LEADS TO LM25 · LM31

TIME $\approx$ 3 h + lab

LEARNING OUTCOMES

Mastery *After this module you should be able to:*

☐	1 derive the sums of arithmetic and geometric sequences, and state when an infinite geometric series converges
☐	2 calculate present and future values of single sums, ordinary annuities and annuities due, deriving each formula from the geometric series
☐	3 value perpetuities, growing annuities and growing perpetuities, and calculate a stock's implied required return and implied growth rate with the Gordon growth model
☐	4 price zero-coupon, coupon and amortizing bonds, and solve for yield to maturity and internal rate of return numerically
☐	5 construct a loan amortization schedule, and compare level-payment, equal-principal and grace-period repayment
☐	6 apply cash-flow additivity and no-arbitrage to derive implied forward rates, forward exchange rates and the fair value of an index future
☐	7 decompose the TX-TAIEX basis into carry, dividends and a residual, and calculate the dividend yield and financing rate it implies
☐	8 implement time-value-of-money calculations in Python

LEARNING MODULE OVERVIEW

- Every annuity, level-payment loan, bond and dividend-discount formula in this module is the sum of a geometric series, $a_1(1 - x^n)/(1 - x)$. A discount factor sits in the ratio. An infinite series converges only when $|x| < 1$, which is why the Gordon model needs $\gamma < k$.
- An annuity due is worth exactly $1 + i$ times the matching ordinary annuity. At a 2% discount rate a perpetuity collects only 45% of its value in its first 30 years; at 8% it collects 90%.
- The Gordon model reads a price as an implied return, $k = D_1/P_0 + \gamma$. On 8 October 2026 0050's trailing yield was 1.39%. With its 10-year dividend growth of 8.43% the implied required return is 9.94%, and two thirds of the model value lies beyond year 30.
- A bond's price falls strictly as its yield rises, so every price has exactly one yield to maturity. Bisection halves the bracket around it at each step and pins it down to 10^{-8} in 22 halvings.
- In a level-payment loan the principal portions grow geometrically at $1 + i$ and the balance equals the present value of the payments still due. On an illustrative NT\$10 million, 30-year mortgage at 2.2%, equal-principal repayment pays NT\$360,064 less interest (undiscounted) and a 3-year grace period NT\$261,984 more. At the loan rate every schedule is worth exactly the principal.
- No-arbitrage turns spot rates into forward rates, interest differentials into forward exchange rates, and the index into a futures fair value $F = Se^{(r_f - q)\tau}$.
- On 27 June 2025, the most extreme session of the sample, the December TX contract settled 880 points below TAIEX. It implied dividends of $q\tau = 4.92\%$ before expiry; those that actually went ex came to 1.87%. Dividends explain the summer shape of the discount. Carry (+215 points) and dividends (-422) leave a residual of -673 (-461 on average over 23 June-3 July) that frictionless arbitrage would have removed.

1

INTRODUCTION

A payment has a date as well as an amount: NT\$1 next year is worth less than NT\$1 today, because today's money earns interest. Discounting at a constant rate multiplies each payment by the same factor per period, so level or steadily growing payments form a **geometric series**. The sum of that series prices deposits, mortgages, bonds and dividend streams. Add one more idea, no-arbitrage, and futures can be priced too.

MOTIVATING CASE Why does TX trade below TAIEX every summer?

At the close on Friday 27 June 2025, TAIEX stood at 22,580.08. Every TX futures contract settled below it: July by 458 points, December by 880 and June 2026 by 1,087 (Exhibit 1). For the December contract this was the deepest discount of the 146 sessions from April to October.

Over June the near-month contract averaged 192 points below the index, against 65 points in April. By September its discount (逆價差) had vanished on average (+0.5 points), and in October TX traded above TAIEX on average (+67 points).

Interest says the opposite. A futures buyer keeps in the bank the cash a stock buyer must spend. With the central bank's discount rate at 2%, futures should sit *above* the index unless dividends going ex before expiry outweigh the interest.

The cost-of-carry formula $F = Se^{(r_f - q)\tau}$ turns each price into the dividend yield q the market must expect. At $r_f = 2\%$ the December contract implied dividends of $q\tau = 4.92\%$ before its expiry on 17 December. The dividends that went ex in the whole of calendar 2025 came to 2.93%.

Is the discount a dividend forecast, a bearish bet or a mispricing? Section 8 splits each contract's discount into carry, dividends and a residual, measured with TWSE's total-return index. It checks how much rests on one extreme day.

Exhibit 1: TX futures against TAIEX on 27 June 2025

Contract	Expiry	Days	Volume (lots)	Settlement	Basis (points)	Implied $q\tau$ ($r_f = 2\%$)
Jul 2025	2025-07-16	19	60,785	22,122	-458.08	2.15%
Aug 2025	2025-08-20	54	1,436	21,948	-632.08	3.14%
Sep 2025	2025-09-17	82	22	21,834	-746.08	3.81%
Dec 2025	2025-12-17	173	49	21,700	-880.08	4.92%
Mar 2026	2026-03-18	264	9	21,607	-973.08	5.85%
Jun 2026	2026-06-17	355	2	21,493	-1,087.08	6.88%

Note: TAIEX close 22,580.08. Days = calendar days to expiry; volume = contracts traded in the regular session; settlement = daily settlement price F ; basis = $F - S$. Implied dividends $q\tau = r_f\tau - \ln(F/S)$, derived in Section 7, with $\tau = \text{days}/365$ (actual/365, as DBS Taiwan accrues NTD deposits; LM1). The last trades of the September, March and June 2026 contracts were 34, 16 and 54 points below their settlements. Source: FinMind [TaiwanFuturesDaily](#) (TX, regular session) and [TaiwanStockPrice](#) (TAIEX), retrieved 10 October 2026; expiry: the third Wednesday of the delivery month (TAIFEX contract specification).

Sections 2–6 build the geometric sum into annuities, perpetuities, the Gordon model, bonds and loans. Section 7 adds no-arbitrage and the futures fair value used above; Section 8 resolves the case, and Section 9 collects the code.

NOTE Notation in this module

i is a rate per period, $v := 1/(1+i)$ the discount factor and n the number of periods. j, t are indices and m, N counts of periods. A sequence has first term a_1 , partial sums $\mathcal{S}_n$ and difference d (arithmetic) or ratio x (geometric). PMT is a level payment and γ the simple growth rate of cash flows per period (the book's g is a log growth rate).

y is a yield to maturity or IRR, z_t the spot rate for maturity t and y_{t_1, t_2} the forward rate from t_1 to t_2 . A stock has required return k and next-year earnings E_1 ; a bond has face value M and coupon C . B_t is the balance of an account or loan after t periods, B_0 at the start (no Brownian motion here). Loan payment t splits into interest I_t and principal Q_t .

In Sections 7–8, r_f and q are the continuously compounded financing rate and dividend yield, and r_b, r_ℓ the borrowing and short-sale rates. δ is the fraction of the index going ex on a day and TR the total-return index. S is the index (or a spot exchange rate) and F a futures or forward price; $F - S$ is the **basis** and ε its residual. τ is a span in years; for a futures contract $\tau = T - t$, from the pricing date t to the expiry T .

2 SEQUENCES AND GEOMETRIC SERIES

- ☐ derive the sums of arithmetic and geometric sequences, and state when an infinite geometric series converges

A deposit earning simple interest grows by the same amount each year; one earning compound interest grows by the same factor. The first is an arithmetic sequence, the second a geometric one.

DEFINITION Arithmetic and geometric sequences

A sequence $a_1, a_2, a_3, \dots$ is **arithmetic** with common difference d if $a_{j+1} = a_j + d$, so that $a_j = a_1 + (j-1)d$. It is **geometric** with common ratio x if $a_{j+1} = x a_j$, so that $a_j = a_1 x^{j-1}$. A **series** is the sum of the terms of a sequence.

NT\$100 at 2% simple interest is worth 100, 102, 104, ... after 0, 1, 2, ... years: an arithmetic sequence with $d = 2$. At 2% compound interest it is worth 100, 102, 104.04, ...: a geometric sequence with $x = 1.02$. The discount factors $v, v^2, v^3, \dots$ that convert future money into present money are geometric with ratio $v = 1/(1+i) < 1$.

Summing an arithmetic sequence

Write the sum forwards and backwards and add the two lines term by term:

$$\begin{aligned}\mathcal{S}_n &= a_1 + a_2 + \dots + a_n, \\ \mathcal{S}_n &= a_n + a_{n-1} + \dots + a_1.\end{aligned}$$

Each pair $a_j + a_{n+1-j} = 2a_1 + (n-1)d = a_1 + a_n$ has the same sum, and there are n pairs, so

$$\sum_{j=1}^n a_j = \frac{n(a_1 + a_n)}{2}. \quad (6.1)$$

Section 6 uses (6.1) for equal-principal loans, whose payments fall arithmetically.

Summing a geometric sequence

Let $\mathcal{S}_n = a_1 + a_1x + \cdots + a_1x^{n-1}$. Multiplying by x shifts every term one place, $x\mathcal{S}_n = a_1x + \cdots + a_1x^n$, and subtracting cancels all but two terms: $\mathcal{S}_n - x\mathcal{S}_n = a_1 - a_1x^n$.

RESULT Geometric sums

For $x \neq 1$,

$$\sum_{j=0}^{n-1} a_1x^j = a_1 \frac{1-x^n}{1-x}, \quad (6.2)$$

and for $x = 1$ the sum is na_1 . If $|x| < 1$, the infinite series converges:

$$\sum_{j=0}^{\infty} a_1x^j = \frac{a_1}{1-x}. \quad (6.3)$$

If $|x| \geq 1$ and $a_1 \neq 0$, the infinite series diverges.

Convergence follows from the gap between the full sum and a partial sum,

$$\frac{a_1}{1-x} - \mathcal{S}_n = \frac{a_1x^n}{1-x}.$$

For $|x| < 1$, $|x|^n = e^{n \ln |x|} \rightarrow 0$ because $\ln |x| < 0$ (LM1), so the gap closes. For $|x| \geq 1$ the terms a_1x^j do not shrink to zero. A series whose terms do not vanish cannot settle on a finite sum (Stewart, Clegg and Watson 2021).

By the gap formula, the fraction of an infinite geometric sum beyond its first n terms is exactly x^n . For $x = v$ this is the share of a perpetuity's value paid after year n .

EXAMPLE 1

How much of a perpetuity is paid in the first 30 years?

A security pays NT\$1 at the end of every year forever. At discount rates of 2%, 5% and 8%, compute its value and the share of it paid within 10, 30 and 50 years.

SOLUTION

The payments form a geometric series with first term v and ratio v . By (6.3) the value is $v/(1 - v) = 1/i$: NT\$50.00 at 2%, NT\$20.00 at 5% and NT\$12.50 at 8%. By the gap formula the share paid after year n is v^n , so the share paid within n years is $1 - v^n$:

Rate	Value	Within 10 years	Within 30 years	Within 50 years
2%	50.00	18.0%	44.8%	62.8%
5%	20.00	38.6%	76.9%	91.3%
8%	12.50	53.7%	90.1%	97.9%

At 8%, 90% of the value arrives within 30 years; at 2%, less than half does. Low discount rates make distant cash flows matter, so valuations at low rates hinge on assumptions about the far future (Section 4).

Every time-value problem is a recurrence

An account that earns i per period and receives a net cash flow CF_t at the end of period t evolves as $B_t = (1 + i)B_{t-1} + CF_t$. Deposits have $CF_t > 0$; withdrawals and loan payments have $CF_t < 0$. Unrolling the recursion from B_0 gives

$$B_n = (1 + i)^n B_0 + \sum_{t=1}^n (1 + i)^{n-t} CF_t, \quad (6.4)$$

as substituting (6.4) for B_{n-1} into the recursion confirms. Level cash flows make the sum geometric with ratio $1 + i$; flows growing at rate γ make it geometric with ratio $(1 + \gamma)/(1 + i)$ after factoring.

KNOWLEDGE CHECK 1

In the 52-week savings challenge (52週存錢法) a saver deposits NT\$100 in week 1, NT\$200 in week 2, ..., NT\$5,200 in week 52. How much is deposited in total? A variant deposits NT\$100 in week 1 and then, each week, 95% of the previous week's amount, forever. What is its total?

ANSWER

The first plan is arithmetic: by (6.1), $52 \times (100 + 5,200)/2 = \text{NT\$}137,800$. The second is geometric with ratio 0.95: by (6.3), $100/(1 - 0.95) = \text{NT\$}2,000$. A ratio below 1 caps an infinite sum; a constant increment does not.

3

PRESENT VALUE, FUTURE VALUE AND ANNUITIES

- calculate present and future values of single sums, ordinary annuities and annuities due, deriving each formula from the geometric series

One period at rate i turns PV into $(1 + i)PV$, and n periods compound to

$$FV_n = PV(1 + i)^n, \quad PV = FV_n(1 + i)^{-n} = FV_n v^n.$$

The rate must match the period; LM1 converts quoted annual rates into periodic and effective rates.

Discounting multiplies each cash flow by a number, so a stream's present value is the sum of its parts' present values, $PV = \sum_t CF_t v^t$. This **cash-flow additivity** lets us cut a complicated stream into simple pieces, value each piece and add. Section 7 turns the same principle into no-arbitrage pricing.

Ordinary annuities

An **ordinary annuity** pays PMT at the end of each of n periods, at $t = 1, \dots, n$. Its present value is a geometric series with first term $PMT v$ and ratio v , so by (6.2)

$$PV = PMT \sum_{t=1}^n v^t = PMT v \frac{1 - v^n}{1 - v}.$$

Since $1 - v = i/(1 + i) = iv$, the factor $v/(1 - v)$ equals $1/i$, and

$$PV_{\text{ord}} = PMT \frac{1 - (1 + i)^{-n}}{i}. \quad (6.5)$$

Compounding (6.5) forward n periods gives the value on the date of the last payment:

$$FV_{\text{ord}} = PMT \frac{(1 + i)^n - 1}{i}. \quad (6.6)$$

We write $a_{n,i}$ and $s_{n,i}$ for the two fractions, the **annuity factors**.

Annuities due

An **annuity due** pays at the start of each period, at $t = 0, \dots, n - 1$ (Exhibit 2). Every payment arrives one period earlier than in the ordinary annuity, so every value is multiplied by $1 + i$:

$$PV_{\text{due}} = (1 + i) PV_{\text{ord}}, \quad FV_{\text{due}} = (1 + i) FV_{\text{ord}}. \quad (6.7)$$

Rent, insurance premiums and leases are usually paid in advance (annuities due); loan payments and bond coupons are paid in arrears (ordinary annuities).

Exhibit 2: Ordinary annuity and annuity due on a timeline

Time t	0	1	2	...	$n - 1$	n
Ordinary annuity		PMT	PMT	...	PMT	PMT
Annuity due	PMT	PMT	PMT	...	PMT	
Discount factor	1	v	v^2	...	v^{n-1}	v^n

Note: the annuity due is the ordinary annuity shifted one period earlier.

Solving for the payment, the term or the rate

Each of (6.5)–(6.7) links a value, the payment PMT , the rate i , the term n and the timing convention. Given all but one, the last follows. The payment enters linearly. The term solves with logarithms: from (6.5), $v^n = 1 - PV\ i/PMT$, so

$$n = -\frac{\ln(1 - PV\ i/PMT)}{\ln(1 + i)},$$

which requires $PMT > PV\ i$.

A payment no larger than the interest $PV\ i$ never reduces the balance, so the payments can continue forever (Section 4). The rate has no closed form in general; Section 5 finds it numerically.

EXAMPLE 2

A down payment and a retirement pot

- (a) A couple wants NT\$3,000,000 for a down payment in 8 years. Their account pays 2.4% a year, credited monthly at 0.2% (illustrative). What deposit at the end of each month reaches the target?
- (b) A retiree holds NT\$10,000,000 in an account paying 0.25% a month (3% a year, illustrative), withdrawing NT\$60,000 at the end of each month. How long does the money last?

SOLUTION

- (a) Solve (6.6) for PMT with $i = 0.002$ and $n = 96$: $PMT = 3,000,000 \times 0.002 / (1.002^{96} - 1) = \text{NT\$}28,377.08$. The couple deposits NT\$2,724,200 in total, and interest supplies the remaining NT\$275,800. Depositing at the start of each month instead, an annuity due, divides the deposit by 1.002, to NT\$28,320.44.
- (b) Here $PV\ i/PMT = 10,000,000 \times 0.0025/60,000 = 0.41667$, so $n = -\ln(0.58333)/\ln(1.0025) = 215.87$ months, about 18.0 years. Without interest the money would last $10,000,000/60,000 = 166.7$ months (13.9 years). A withdrawal of NT\$25,000, equal to the monthly interest, would last forever.

PITFALL Timing and period mismatches

Two errors account for most wrong annuity answers. The first is timing: $a_{n,i}$ assumes the first payment one period from now, so payments in advance need the extra factor $1 + i$. The second is the period: monthly payments need a monthly rate and a count of months. Under the nominal-rate convention, 2.4% a year over eight years means $i = 0.2\%$ and $n = 96$, not $i = 2.4\%$ and $n = 8$.

導讀 一條等比級數，走遍所有現金流

折現就是每過一期乘上 $v = 1/(1 + i)$ ，所以一串固定金額的現金流，現值必然是等比級數：首項 $PMT v$ 、公比 v ，套用 (6.2) 就得到年金公式。期初年金（房租、保費）只是把整串現金流往前移一期，所以剛好乘上 $1 + i$ 。級數的「尾巴」 x^n 告訴你第 n 期之後還藏著多少價值：折現率 8% 時，永續年金九成的價值在 30 年內就收完；折現率 2% 時，30 年只收到 45%。利率越低，越遙遠的未來越重要。

KNOWLEDGE CHECK 2

NT\$1,000 is paid each year for 10 years and money earns 3%. Compute the present value and the future value as an ordinary annuity and as an annuity due. By what factor does each annuity-due value exceed the matching ordinary-annuity value?

ANSWER

Ordinary: $PV = 1,000 \times (1 - 1.03^{-10})/0.03 = 8,530.20$ and $FV = 1,000 \times (1.03^{10} - 1)/0.03 = 11,463.88$. Due: 8,786.11 and 11,807.80. Each due value is the ordinary value times 1.03, by (6.7).

4

PERPETUITIES, GROWTH AND THE GORDON MODEL

- ☐ value perpetuities, growing annuities and growing perpetuities, and calculate a stock's implied required return and implied growth rate with the Gordon growth model

Let the number of payments grow without bound, or let the payments themselves grow. Both limits are geometric series, and together they value preferred shares, common stocks and the terminal stage of any long-lived cash flow.

Perpetuities

As $n \rightarrow \infty$ in (6.5), $(1 + i)^{-n} \rightarrow 0$ and the value of a **perpetuity** appears:

$$PV_{\text{perp}} = \frac{PMT}{i}. \quad (6.8)$$

A perpetual preferred share paying NT\$2.40 a year is worth NT\$60.00 at a required return of 4%. If the first payment comes at $t = m + 1$ instead of $t = 1$, the stream is worth PMT/i at $t = m$ and $v^m PMT/i$ today. A first dividend in six years ($m = 5$) gives NT\$49.32. A perpetuity due, which also pays today, is worth $(1 + i) PMT/i = NT\$62.40$.

Growing annuities and perpetuities

Let the payment at t be $PMT_1(1 + \gamma)^{t-1}$, growing at rate γ per period. Factoring out $PMT_1/(1 + i)$,

$$PV = \sum_{t=1}^n \frac{PMT_1(1 + \gamma)^{t-1}}{(1 + i)^t} = \frac{PMT_1}{1 + i} \sum_{t=0}^{n-1} \left(\frac{1 + \gamma}{1 + i} \right)^t,$$

a geometric series with ratio $x = (1 + \gamma)/(1 + i)$. Since $1 - x = (i - \gamma)/(1 + i)$, (6.2) gives

$$PV_{\text{grow}} = \frac{PMT_1}{i - \gamma} \left[1 - \left(\frac{1 + \gamma}{1 + i} \right)^n \right], \quad \gamma \neq i. \quad (6.9)$$

When $\gamma = i$ every term equals $PMT_1/(1 + i)$ and the sum is $n PMT_1/(1 + i)$. As $n \rightarrow \infty$, (6.3) applies if and only if $x < 1$, that is $\gamma < i$:

$$PV_{\text{grow perp}} = \frac{PMT_1}{i - \gamma}, \quad \gamma < i. \quad (6.10)$$

If $\gamma \geq i$ the series diverges: payments that grow at least as fast as the discount rate have no finite value. As a check, $PMT_1 = 1$, $i = 5\%$, $\gamma = 3\%$ and $n = 20$ give 15.9648, both by (6.9) and by summing the 20 terms. With $\gamma = i$ the value is $20/1.05 = 19.0476$.

The Gordon growth model

A share entitles its holder to dividends $D_1, D_2, \dots$. If they grow at a constant rate γ forever and investors require a return $k > \gamma$, (6.10) prices the share:

$$P_0 = \frac{D_1}{k - \gamma} = \frac{D_0(1 + \gamma)}{k - \gamma}. \quad (6.11)$$

This is the **Gordon growth model** (Gordon 1959). Read backwards, a market price implies a required return or a growth rate. Solving (6.11) for k gives the **implied required return**, dividend yield plus growth:

$$k = \frac{D_1}{P_0} + \gamma. \quad (6.12)$$

Solving instead for γ , $kP_0 - \gamma P_0 = D_0 + \gamma D_0$ gives the **implied growth rate**

$$\gamma = \frac{kP_0 - D_0}{P_0 + D_0}. \quad (6.13)$$

Dividing (6.11) by next year's earnings E_1 gives the **justified forward P/E**, $P_0/E_1 = (D_1/E_1)/(k - \gamma)$: the payout ratio divided by $k - \gamma$. A 40% payout with $k = 9\%$ and $\gamma = 4\%$ justifies a forward P/E of 8.0.

Changing growth

A company may grow fast for some years and then mature. A **two-stage model** values the first N dividends with (6.9) and the rest, growing at γ_2 , with (6.11) applied at $t = N$:

$$P_0 = \sum_{t=1}^N \frac{D_t}{(1+k)^t} + \frac{1}{(1+k)^N} \cdot \frac{D_{N+1}}{k - \gamma_2}.$$

EXAMPLE 3

A two-stage dividend model

A company has just paid a dividend of NT\$10. Analysts expect 12% growth for five years and 4% a year thereafter; the required return is 9% (illustrative). Value the share.

SOLUTION

Stage 1: $D_1, \dots, D_5 = 11.20, 12.544, 14.0493, 15.7352, 17.6234$. Their present value at 9% is 54.2831, by discounting each one or by (6.9) with $PMT_1 = 11.20$, $i = 9\%$, $\gamma = 12\%$ and $n = 5$. The formula works with $\gamma > i$ because the stream is finite.

Stage 2: $D_6 = D_5 \times 1.04 = 18.32835$ (unrounded $D_5 = 17.623417$), so $P_5 = 18.32835/(0.09 - 0.04) = 366.567$, worth $366.567/1.09^5 = 238.2434$ today.

Hence $P_0 = 54.2831 + 238.2434 = \text{NT\$}292.53$, of which the terminal value is 81.4%. A single-stage model at 4% would give NT\$208.00; at 12% it would give no answer, because $\gamma > k$.

What the price of 0050 implies

An index fund is a claim on the dividends of its basket, so the Gordon model can be read off its price. Exhibit 3 sets out the inputs for 0050 ten years apart.

Exhibit 3: Gordon-model inputs for 0050, ten years apart

	7 October 2016	8 October 2026
Price (split-adjusted, NT\$)	17.91	114.95
Trailing 12-month cash dividends (NT\$)	0.7125	1.6000
Dividend yield D_0/P_0	3.98%	1.39%
Implied growth at $k = 8\%$, eq. (6.13)	3.87%	6.52%
Implied growth at $k = 10\%$	5.79%	8.49%

Note: closing prices; 2016 values are divided by 4 for the 4-for-1 split of 18 June 2025. Dividends: NT\$2.00 (26 October 2015) and NT\$0.85 (28 July 2016) before the split; NT\$1.00 (22 January 2026) and NT\$0.60 (21 July 2026) after it. Over ten annual dividend periods dividends grew $(1.6000/0.7125)^{1/10} - 1 = 8.43\%$ a year; the price grew 20.43% a year. Source: FinMind [TaiwanStockPrice](#) and [TaiwanStockDividendResult](#), retrieved 10 October 2026.

EXAMPLE 4

The required return priced into 0050

On 8 October 2026, 0050 closed at NT\$114.95 with trailing cash dividends of NT\$1.60, after a decade of 8.43% annual dividend growth. (a) What required return does the price imply if that growth continues? (b) What growth does it imply at a required return of 8%? (c) What share of the Gordon value comes from dividends paid after year 30?

SOLUTION

(a) By (6.12), $k = 1.60 \times 1.0843/114.95 + 0.0843 = 1.51\% + 8.43\% = 9.94\%$.

(b) By (6.13), $\gamma = (0.08 \times 114.95 - 1.60)/(114.95 + 1.60) = 7.596/116.55 = 6.52\%$.

(c) The share after year n is x^n with $x = (1 + \gamma)/(1 + k)$: $(1.0843/1.0994)^{30} = 0.66$. Two thirds of the value lies beyond year 30, against 24% for a stock with $k = 9\%$ and $\gamma = 4\%$.

The model is fragile here. Holding D_1 fixed, lowering $k - \gamma$ by half a percentage point raises the model price from NT\$114.95 to NT\$171.90. Raising it by half a point lowers it to NT\$86.34.

Between 2016 and 2026 the yield fell from 3.98% to 1.39%. At a constant 10% required return the implied growth rose from 5.79% to 8.49% (Exhibit 3). Either investors came to expect faster dividend growth or they accepted a lower return: the model identifies only $k - \gamma = D_1/P_0$, never its two parts.

PITFALL Gordon near its edge

Formula (6.11) sums a geometric series with ratio $(1 + \gamma)/(1 + k)$, so as γ approaches k the price becomes hypersensitive to both inputs. Report a range of (k, γ) pairs, not a point. Never use price growth as γ : 0050's price grew 20.43% a year over the decade, and any $\gamma \geq k$ leaves the sum infinite. Put $D_1 = D_0(1 + \gamma)$, not D_0 , in the numerator, or the price is understated by the factor $1 + \gamma$.

導讀 股利殖利率就是 $k - \gamma$

高登模型 $P_0 = D_1/(k - \gamma)$ 說：預期股利殖利率 D_1/P_0 剛好等於「要求報酬率減成長率」。0050 在 2026/10/8 的近 12 個月股利殖利率 D_0/P_0 只有 1.39%，換成下一年的股利， $D_1/P_0 = k - \gamma$ 也只有 1.51%，代表市場定價隱含的 k 與 γ 非常接近：若股利維持過去十年 8.43% 的成長，隱含要求報酬率約 9.94%。但 $k - \gamma$ 很小時，公比 $(1 + \gamma)/(1 + k)$ 接近 1，級數收斂極慢，30 年後的股利占了價值的三分之二； $k - \gamma$ 只要變動 0.5 個百分點，模型價格就從 114.95 跳到 171.90，或掉到 86.34。模型只能告訴你兩者的「差」，分不出是成長預期變高，還是要求報酬變低。

KNOWLEDGE CHECK 3

A perpetual preferred share pays NT\$3.00 a year and trades at NT\$62.50. What required return does the price imply? At that return, what would the share be worth if the issuer raised the dividend by 1% a year forever, from NT\$3.00 next year?

ANSWER

By (6.8), $i = 3.00/62.50 = 4.8\%$. By (6.10), $3.00/(0.048 - 0.01) = \text{NT\$}78.95$. One percentage point of growth adds 26% to the price, because it cuts the denominator from 4.8% to 3.8%.

5

BOND PRICING AND YIELD TO MATURITY

- ☐ price zero-coupon, coupon and amortizing bonds, and solve for yield to maturity and internal rate of return numerically

A bond's price is the present value of its promised cash flows, so bond pricing is annuity pricing. The only new problem is solving for the rate.

Three kinds of bond

A **zero-coupon bond** pays only its face value M at maturity, so its price is $P = M(1 + y)^{-n}$: it is a discount instrument. A **coupon bond** pays a coupon C each period (coupon rate C/M) and M at maturity: an ordinary annuity plus a single sum,

$$P = C \frac{1 - (1 + y)^{-n}}{y} + M(1 + y)^{-n}. \quad (6.14)$$

An **amortizing bond** pays a level amount PMT that covers interest and repays principal over its life. It is worth PMT $a_{n,y}$: the loan of Section 6 seen from the lender's side.

Rearranging (6.14) shows when a bond trades above par. Because $yM a_{n,y} + M(1 + y)^{-n} = M$, a bond whose coupon rate equals its yield is worth exactly par, and subtracting gives

$$P - M = (C - yM) a_{n,y}. \quad (6.15)$$

Price exceeds par exactly when the coupon rate exceeds the yield, by the present value of the coupon surplus.

A coupon bond is also a portfolio of zero-coupon bonds, one per cash flow. By cash-flow additivity each cash flow must be discounted at its own **spot rate** z_t :

$$P = \sum_{t=1}^n \frac{CF_t}{(1 + z_t)^t}.$$

The **yield to maturity** y is the single rate that reproduces this price in (6.14). It summarizes the spot curve, weighted by the bond's cash flows.

EXAMPLE 5

Three bonds at the same yield

Price three 5-year bonds with face 100 and annual payments at a yield of 2% (illustrative). (a) A zero-coupon bond; (b) a 3% coupon bond; (c) an amortizing bond with a 3% coupon rate, repaid in five equal payments.

SOLUTION

(a) $100/1.02^5 = 90.5731$.

(b) With $a_{5,2\%} = 4.71346$, (6.14) gives $3 \times 4.71346 + 90.5731 = 104.7135$. By (6.15) the premium is $(3 - 2) \times 4.71346 = 4.7135$: one point of excess coupon a year for five years, discounted.

(c) The level payment that repays 100 at 3% is $100 \times 0.03/(1 - 1.03^{-5}) = 21.8355$, by the loan formula (6.16) of Section 6. At a 2% yield the bond is worth $21.8355 \times 4.71346 = 102.921$. Its premium is smaller than in (b) because principal comes back early, so the 3% coupon is earned on a shrinking balance.

Yield to maturity as an implied return

Given a market price, the yield to maturity solves $P(y) = P_{\text{mkt}}$, a polynomial equation of degree n in $1/(1+y)$. It has no closed-form solution in general but is well behaved. Every term of $P(y)$ falls as y rises, so P is continuous and strictly decreasing. It falls from $\sum_t CF_t$ at $y = 0$ towards 0 as $y \rightarrow \infty$, so any price in between has exactly one positive yield.

Bisection exploits this monotonicity. Start with a bracket $[y_{\text{lo}}, y_{\text{hi}}]$ on which $P(y) - P_{\text{mkt}}$ changes sign, and repeatedly keep the half whose ends still differ in sign. Each step halves the bracket and with it the bound on the error.

The error itself need not fall: it rises from step 3 to step 4 in Exhibit 4. After j halvings the midpoint lies within $(y_{\text{hi}} - y_{\text{lo}})/2^{j+1}$ of the yield. From a bracket 5 percentage points wide, 10^{-8} takes 22 halvings and 23 midpoints, since $2^{23} \geq 0.05/10^{-8} > 2^{22}$. Newton's method, which also uses the slope $P'(y)$, needs far fewer (LM4).

Exhibit 4: Bisection for the yield of a 10-year, 1.5% annual-coupon bond priced at 97.20

Step	Lower end	Upper end	Midpoint	$P(\text{mid}) - 97.20$
1	0.000000%	5.000000%	2.500000%	-5.9521
2	0.000000%	2.500000%	1.250000%	+5.1364
3	1.250000%	2.500000%	1.875000%	-0.5906
4	1.250000%	1.875000%	1.562500%	+2.2255
5	1.562500%	1.875000%	1.718750%	+0.8058
6	1.718750%	1.875000%	1.796875%	+0.1047
7	1.796875%	1.875000%	1.835938%	-0.2437
8	1.796875%	1.835938%	1.816406%	-0.0696
⋮				
23			1.808601%	0.0000

Note: illustrative bond. $P(0) = 115.00$ and $P(5\%) = 72.97$ bracket the price. A positive value means the trial yield is too low, so the root lies in the upper half. Step 23 takes the midpoint of the bracket left after 22 halvings, 1.2×10^{-8} wide.

EXAMPLE 6

The implied return of a savings policy

A savings-type insurance policy (儲蓄險) costs NT\$100,000 a year, paid at the start of each of six years ($t = 0, \dots, 5$). It pays NT\$700,000 at the end of year 10 (illustrative terms; ignore the insurance cover). What annual return does it offer, and how does it compare with a deposit paying 1.7%?

SOLUTION

The internal rate of return (IRR) y sets the net present value (NPV) to zero:

$$-100,000 \sum_{t=0}^5 (1+y)^{-t} + 700,000 (1+y)^{-10} = 0.$$

At $y = 0$ the NPV is +100,000; at 10% it is negative. Bisection on $[0\%, 10\%]$ tries 5.000% (NPV -103,208), 2.500% (-17,744), 1.250% (+36,443), 1.875% (+8,268) and 2.1875% (-4,998). It reaches $y = 2.0683\%$, to within 10^{-8} , at the 24th midpoint.

At 1.7% the NPV is +NT\$15,932. The six premiums deposited at 1.7% would grow to NT\$681,143 by year 10, NT\$18,857 short of the policy's payout. The comparison assumes a deposit rate of 1.7% for ten years and a policy held to maturity; it ignores surrender values.

Is an IRR always unique? Write the NPV as a polynomial in $v = 1/(1+y)$. Descartes' rule of signs, stated here without proof (Wang 2004), counts its positive roots. There are as many, counted with multiplicity, as the sign changes among its non-zero coefficients, or fewer by an even number.

Outflows followed by inflows, as in Example 6, change sign once: one IRR. The flows -100 , $+230$ and -132 at $t = 0, 1, 2$ change sign twice, and the IRR is both 10% and 20%. For such streams, report the NPV at a stated rate instead.

Pull to par

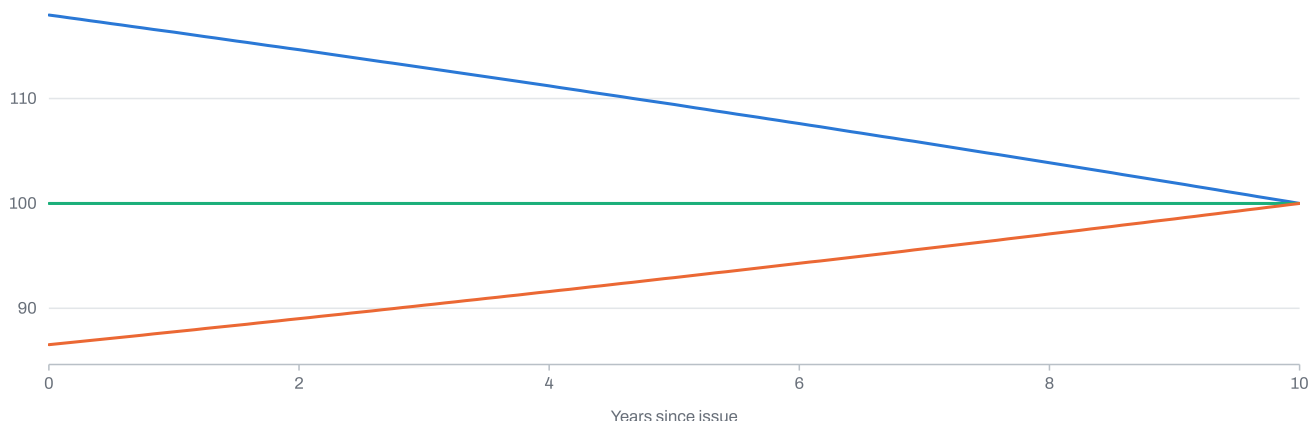
Holding the yield fixed, a bond's price still changes as time passes, because the remaining cash flows draw nearer. Exhibit 5 shows three 10-year bonds priced at a constant 2% yield, all converging to par at maturity. The premium bond (4% coupon) falls from 117.97 and the discount bond (0.5% coupon) rises from 86.53.

The par bond is at 100 on coupon dates; between them its clean price dips to 99.995. Futures converge too: at expiry a futures price meets its final settlement price (Section 7).

Exhibit 5: Pull to par at a constant 2% yield

— 4% (premium) coupon — 2% (par) coupon — 0.5% (discount) coupon

Clean price per 100 face



Note: illustrative 10-year bonds with annual coupons of 4%, 2% and 0.5%; clean prices (full price minus accrued coupon), monthly. After five years the premium bond is worth 109.43 and the discount bond 92.93.

PITFALL Yield to maturity is a quote, not a forecast

The yield to maturity is the IRR of the promised cash flows at today's price. An investor earns it only by holding to maturity, receiving every payment and reinvesting every coupon at that same yield. Sell early, reinvest at other rates or suffer a default, and the realized return differs. The same caveat applies to the IRR of Example 6.

導讀 到期殖利率 = 隱含報酬率 = IRR

債券價格就是「年金 + 到期本金」的現值。反過來，由市價解出的折現率就是到期殖利率，和儲蓄險的 IRR 是同一件事：讓淨現值等於零的那個利率。因為價格隨到期殖利率嚴格遞減，一般債券的到期殖利率存在而且唯一。二分法每一步把包住答案的區間砍半，誤差的上限也跟著減半，但實際誤差不一定每步變小；從 5 個百分點寬的區間出發，砍半 22 次、取第 23 個中點，就能精確到 10^{-8} 。到期殖利率只是報價：只有持有到期、每筆票息都以同一個殖利率再投資，才會真的賺到它。

KNOWLEDGE CHECK 4

Without computing, is a 3-year bond with a 2% annual coupon priced above or below par at a 2.5% yield? Confirm with (6.14) and (6.15).

ANSWER

Below par, by (6.15), because the coupon rate is below the yield. $P = 2 \times a_{3,2.5\%} + 100 \times 1.025^{-3} = 98.5720$; equivalently $P - 100 = (2 - 2.5) \times a_{3,2.5\%} = -1.4280$.

6

LOANS AND AMORTIZATION

- ☐ construct a loan amortization schedule, and compare level-payment, equal-principal and grace-period repayment

A level-payment loan is an ordinary annuity seen from the borrower's side. The lender hands over B_0 today in exchange for n payments of PMT, so $B_0 = \text{PMT } a_{n,i}$ and

$$\text{PMT} = \frac{B_0 i}{1 - (1 + i)^{-n}}. \quad (6.16)$$

Balance, interest and principal

Recurrence (6.4) with $\text{CF}_t = -\text{PMT}$ gives the balance after t payments in two equivalent forms:

$$B_t = B_0(1 + i)^t - \text{PMT } s_{t,i} = \text{PMT } a_{n-t,i}. \quad (6.17)$$

The first, retrospective, form accumulates the loan and subtracts the accumulated payments. The second, prospective, form says the balance is the present value of the payments still due. Substituting $B_0 = \text{PMT } a_{n,i}$ into the first form shows they agree:

$$\text{PMT} \frac{(1 - v^n)(1 + i)^t - ((1 + i)^t - 1)}{i} = \text{PMT} \frac{1 - v^{n-t}}{i} = \text{PMT } a_{n-t,i}.$$

Each payment splits into interest on the outstanding balance, $I_t = iB_{t-1}$, and principal $Q_t = \text{PMT} - I_t$. The prospective form gives $iB_{t-1} = \text{PMT}(1 - v^{n-t+1})$, so

$$Q_t = \text{PMT } v^{n-t+1}. \quad (6.18)$$

The principal portions form a geometric sequence with ratio $1 + i$, from $\text{PMT } v^n$ in the first payment to $\text{PMT } v$ in the last. Interest portions fall as principal portions rise, and each pair sums to PMT.

Principal exceeds interest, $Q_t > I_t$, exactly when $v^{n-t+1} > \frac{1}{2}$: when the $n - t + 1$ payments remaining span less than the doubling time $\ln 2 / \ln(1 + i)$. At 2.2% a year money doubles in 378.4 months, longer

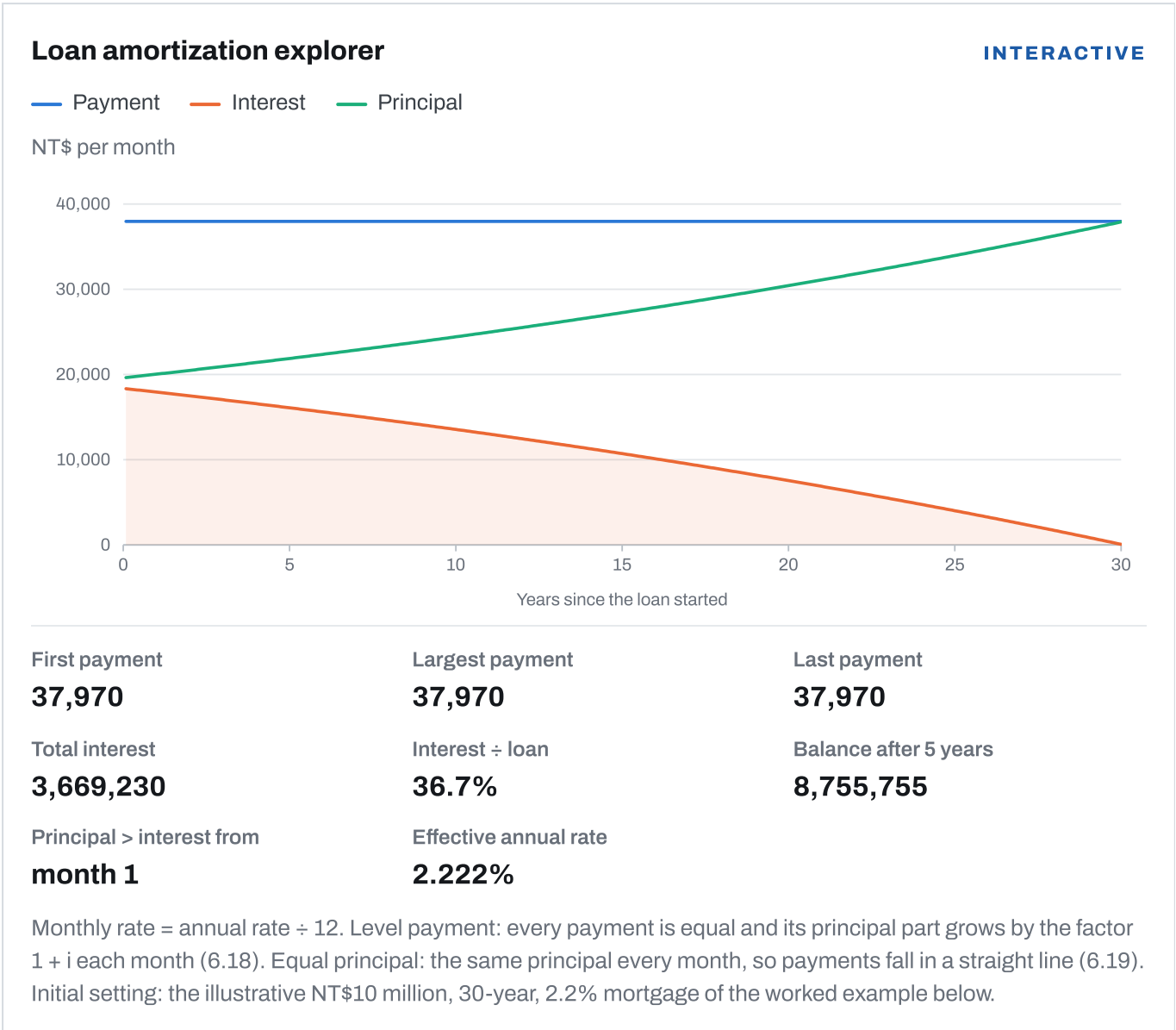
than a 30-year term, so principal exceeds interest from the first payment. At 5% the doubling time is 166.7 months, and principal overtakes interest only at payment 195.

Equal principal and grace periods

Under **equal-principal** repayment the borrower repays B_0/n each period plus interest on the remaining balance. The balance before payment t is $B_0(1 - (t - 1)/n)$, so the payments fall by the constant amount iB_0/n : an arithmetic sequence. Total interest follows from (6.1):

$$\sum_{t=1}^n iB_0\left(1 - \frac{t - 1}{n}\right) = \frac{iB_0}{n} \sum_{j=1}^n j = \frac{iB_0(n + 1)}{2}. \tag{6.19}$$

A **grace period** (寬限期) of m periods charges interest only, iB_0 per period, and then amortizes B_0 over the remaining $n - m$ periods with (6.16).



EXAMPLE 7

An NT\$10 million mortgage, three ways

A buyer borrows NT\$10,000,000 for 30 years at 2.2% a year, paid monthly (illustrative rate; $i = 0.022/12$). Compute (a) the level payment, the split of the first payment and the balance after five years. Compute the payments and total interest (b) under equal-principal repayment and (c) with a three-year grace period.

SOLUTION

(a) By (6.16) with $n = 360$, $PMT = NT\$37,970.08$ (37,970.084143 unrounded, used below). The first payment contains interest of $10,000,000 \times 0.022/12 = 18,333.33$ and principal of 19,636.75, which equals $PMT v^{360}$ by (6.18). After 60 payments the balance is NT\$8,755,754.89 by either form of (6.17): five years of payments, NT\$2,278,205, have repaid only NT\$1,244,245 of principal. Total interest over the life is $360 PMT - B_0 = NT\$3,669,230.29$ (Exhibit 6).

(b) Equal principal means NT\$27,777.78 of principal each month. The first payment is $27,777.78 + 18,333.33 = NT\$46,111.11$, and payments then fall by $iB_0/n = NT\$50.93$ a month to $(B_0/n)(1+i) = NT\$27,828.70$. By (6.19), total interest is $(0.022/12) \times 10,000,000 \times 361/2 = NT\$3,309,166.67$, NT\$360,064 less than in (a). The first payment is 21% higher than in (a); the two payment paths cross at month 161.

(c) With a 36-month grace period the buyer pays NT\$18,333.33 a month for three years, then PMT' from (6.16) with $n = 324$: NT\$40,960.54. Total interest rises to NT\$3,931,214.67, NT\$261,984 more than in (a). Because no principal is repaid for three years, the balance is higher in every month.

At the 2.2% loan rate each schedule is worth exactly NT\$10,000,000; the totals differ only because principal is repaid at different speeds. Repaying faster pays off only if your alternative return is below the loan rate. Discounted at 1.7% a year (0.017/12 a month), equal principal is NT\$63,393 cheaper than level payment and the grace period NT\$49,252 dearer. At 3% the order reverses.

Exhibit 6: An NT\$10 million, 30-year loan at 2.2%: level-payment schedule and three repayment methods

Payment t	Payment	Interest	Principal	Balance after payment
1	37,970.08	18,333.33	19,636.75	9,980,363.25
2	37,970.08	18,297.33	19,672.75	9,960,690.50
120	37,970.08	13,550.88	24,419.20	7,366,971.24
180	37,970.08	10,714.21	27,255.87	5,816,858.62
240	37,970.08	7,548.01	30,422.07	4,086,676.15
359	37,970.08	138.84	37,831.24	37,900.60
360	37,970.08	69.48	37,900.60	0.00

Method	First payment	Payment 37	Last payment	Total interest
Level payment (本息平均攤還)	37,970.08	37,970.08	37,970.08	3,669,230.29
Equal principal (本金平均攤還)	46,111.11	44,277.78	27,828.70	3,309,166.67
Level payment after a 3-year grace period	18,333.33	40,960.54	40,960.54	3,931,214.67

Note: NT\$; illustrative rate, monthly rate $i = 0.022/12$. In every row of the first table, payment = interest + principal; the principal grows by the factor $1 + i$ each month, by (6.18). Total interest is undiscounted.

PITFALL A lower payment is not a cheaper loan

Grace periods and longer terms lower the payment and raise total interest, yet at the loan rate every schedule is worth exactly the principal. Compare loans by effective rate or by the PV of the payments at your own discount rate, not by first payment or undiscounted interest. Loan rates are quoted as nominal annual rates, so the monthly rate is the quote divided by 12. The effective annual rate of a 2.2% loan is $(1 + 0.022/12)^{12} - 1 = 2.223\%$ (LM1).

導讀 本息平均攤還 vs. 本金平均攤還

本息平均攤還的每月付款固定，但其中的本金是公比 $1 + i$ 的等比數列，越後期本金越多；本金平均攤還每月還一樣的本金，付款金額呈等差數列遞減，總利息由 (6.19) 給出。以 1,000 萬元、30 年、利率 2.2%（示意）為例，本金平均攤還少付約 36 萬元利息，但第一期多付 21%；寬限期只繳利息，總利息多出約 26 萬元。這些差額都沒有折現：以貸款利率 2.2% 折現，三種還法的現值都剛好是 1,000 萬元，差別只在本金還得快或慢；只有當資金另作他用的報酬（例如 1.7% 的存款利率）低於貸款利率時，提早還本才划算。無論哪一種方法，貸款餘額永遠等於「剩下各期付款的現值」。

KNOWLEDGE CHECK 5

A NT\$5,000,000 loan runs 20 years at 2.4% a year, paid monthly. Compute the payment and the balance after 10 years. Why is more than half of the principal still outstanding at mid-term?

ANSWER

With $i = 0.2\%$ and $n = 240$, (6.16) gives $\text{PMT} = \text{NT\$}26,252.24$, and (6.17) gives $B_{120} = \text{PMT } a_{120,i} = \text{NT\$}2,798,273$, or 55.97% of the principal. By (6.18) the principal portions grow geometrically at $1 + i$. The first 120 payments carry the smaller portions, so they repay less than half of the loan.

7

CASH-FLOW ADDITIVITY AND NO-ARBITRAGE

- ☐ apply cash-flow additivity and no-arbitrage to derive implied forward rates, forward exchange rates and the fair value of an index future

The **law of one price** says that two portfolios with identical future cash flows must cost the same today. Otherwise buying the cheaper and selling the dearer locks in a riskless profit, an **arbitrage**. With cash-flow additivity it prices anything that can be replicated; this section applies the pair three times.

Implied forward rates

An investor with a two-year horizon can buy a two-year zero at the spot rate z_2 . Alternatively, she can buy a one-year zero at z_1 and agree today to reinvest for the second year at a **forward rate** $y_{1,2}$. Both strategies are riskless and pay off on the same date, so they must grow money equally: $(1 + z_2)^2 = (1 + z_1)(1 + y_{1,2})$. In general, the forward rate y_{t_1,t_2} from year t_1 to year t_2 satisfies

$$(1 + z_{t_2})^{t_2} = (1 + z_{t_1})^{t_1} (1 + y_{t_1,t_2})^{t_2-t_1}. \quad (6.20)$$

With continuous compounding the relation is additive, $y_{t_1,t_2} = (t_2 z_{t_2} - t_1 z_{t_1}) / (t_2 - t_1)$. Taking logs of (6.20) and using $\ln(1 + z_t) \approx z_t$ and $\ln(1 + y_{t_1,t_2}) \approx y_{t_1,t_2}$, which drop second-order terms (LM3), gives the same formula, approximately, for annual rates.

Exhibit 7: Spot rates, discount factors and implied one-year forward rates

Maturity t (years)	1	2	3	4	5
Spot rate z_t	1.50%	1.70%	1.85%	1.95%	2.05%
Discount factor $(1 + z_t)^{-t}$	0.985222	0.966848	0.946492	0.925659	0.903514
Forward rate $y_{t-1,t}$	1.5000%	1.9004%	2.1507%	2.2506%	2.4510%

Note: illustrative curve; $y_{0,1} = z_1$. A 5-year bond with a 3% annual coupon priced off these discount factors is worth 104.5346, a yield to maturity of 2.0369%. On an upward-sloping curve each forward rate exceeds the spot rate of the same end date. The spot rate is a geometric average of the forwards before it.

Forward exchange rates

Quote the exchange rate S as TWD per USD. To hold TWD in τ years, invest TWD 1 at i_{TWD} . Alternatively, convert it into USD $1/S$, invest at i_{USD} and sell forward at F . Both routes are riskless, so with simple annual money-market rates

$$F = S \frac{1 + i_{\text{TWD}} \tau}{1 + i_{\text{USD}} \tau}, \quad (6.21)$$

covered interest parity; with continuous rates, $F = S e^{(r_{\text{TWD}} - r_{\text{USD}})\tau}$. We call $F - S$ the **forward points** and quote them in TWD per USD. The lower-rate currency trades at a forward premium: when TWD rates are below USD rates, a forward buys fewer TWD per USD than spot.

EXAMPLE 8

Forward rates from a spot curve, and a USD/TWD forward

(a) Annual-compounding spot rates for 1 to 5 years are 1.50%, 1.70%, 1.85%, 1.95% and 2.05% (illustrative). Compute the one-year forward rates and the three-year rate starting in two years, $y_{2,5}$.

(b) USD/TWD trades at 30.00 TWD per USD; one-year deposit rates are 1.6% in TWD and 4.0% in USD (illustrative). Compute the no-arbitrage one-year forward rate and the forward points, and show the arbitrage against a dealer quoting 29.50.

SOLUTION

(a) By (6.20), $y_{1,2} = 1.017^2/1.015 - 1 = 1.9004\%$; likewise $y_{2,3} = 2.1507\%$, $y_{3,4} = 2.2506\%$ and $y_{4,5} = 2.4510\%$ (Exhibit 7). For $y_{2,5}$, $(1.0205^5/1.017^2)^{1/3} - 1 = 2.2840\%$; the log approximation $(5 \times 2.05\% - 2 \times 1.70\%)/3 = 2.2833\%$ is off by 0.07 bp.

(b) By (6.21), $F = 30.00 \times 1.016/1.040 = 29.3077$, so the forward points are -0.6923 TWD per USD. The first-order rule $F \approx S(1 + i_{\text{TWD}} - i_{\text{USD}}) = 29.2800$ misses by 0.0277, the dropped term of order $(i_{\text{TWD}} - i_{\text{USD}}) i_{\text{USD}}$. Against a 29.50 quote, borrow TWD 30.00 at 1.6%, buy USD 1, deposit it at 4.0% and sell the USD 1.04 forward at 29.50. At maturity that delivers TWD 30.68 against TWD 30.48 owed: a riskless TWD 0.20.

Index futures and the cost of carry

To replicate an index futures position, borrow $Se^{-q\tau}$ at the continuously compounded rate r_f and buy $e^{-q\tau}$ units of the index basket. Reinvest the dividends, which arrive at the rate q , in the basket. At expiry the position has grown to exactly one unit, worth S_T , and the loan to $Se^{-q\tau}e^{r_f\tau}$. A futures contract bought at F pays $S_T - F$, the same payoff, so the delivery price must be

$$F = Se^{(r_f - q)\tau}. \quad (6.22)$$

If F were higher, an arbitrageur would sell futures and buy the basket (**cash-and-carry**); if lower, buy futures and short the basket (**reverse cash-and-carry**). Futures are marked to market daily, but with a deterministic interest rate their fair value equals that of a forward (Hull 2022). With discrete dividends D_j going ex t_j years after the pricing date ($0 < t_j < \tau$), and treated as received then, the same argument gives

$$F = \left(S - \sum_j D_j e^{-r_f t_j} \right) e^{r_f \tau} = S e^{r_f \tau} - \sum_j D_j e^{r_f(\tau - t_j)}. \quad (6.23)$$

The futures price is the index carried forward at r_f , minus the dividends carried forward to expiry.

The basis $F - S = S(e^{(r_f - q)\tau} - 1) \approx S(r_f - q)\tau$, with a relative error of about $(r_f - q)\tau/2$, is negative whenever $q > r_f$: dividends outweigh interest. With illustrative values $S = 20,000$, $r_f = 2\%$, $q = 3\%$ and $\tau = 0.25$, $F = 19,950.06$ and the basis is -49.94 points, against -50.00 to first order. At expiry TX is cash-settled against the average of the TAIEX values published in the last 30 minutes of trading (TAIFEX). F converges to that average, not to the 13:30 close: the futures version of pull to par.

Reading the fair value backwards

Observed F and S pin down $r_f - q$, not r_f and q separately. Fixing one gives the other:

$$q\tau = r_f\tau - \ln \frac{F}{S}, \quad r_{\text{impl}} = q + \frac{1}{\tau} \ln \frac{F}{S}. \quad (6.24)$$

The first is the **implied dividend yield**, here measured over the contract's life as $q\tau$. The second is the **implied financing rate**, often called the implied repo rate. Two futures with expiries $\tau_1 < \tau_2$ give, by the

logic of (6.20), an implied forward dividend yield between the expiries: $q_{12}(\tau_2 - \tau_1) = r_f(\tau_2 - \tau_1) - \ln(F_2/F_1)$. It does not involve S at all.

Frictions turn (6.22) into a band. Cash-and-carry borrows cash at a rate r_b . Reverse cash-and-carry must borrow every share in the basket, pay lending fees and usually earn less than r_f on the short-sale proceeds. Its effective rate is some $r_\ell < r_b$.

Arbitrage then holds the futures price only within

$$S e^{(r_\ell - q)\tau} \leq F \leq S e^{(r_b - q)\tau}.$$

The lower bound is the weak one, because shorting a whole index basket is harder than buying it.

導讀 無套利：同樣的現金流，同樣的價格

遠期利率、遠期匯率、期貨合理價，背後都是同一個論證：兩個未來現金流完全相同的組合，今天的價格必須相同，否則就能低買高賣、無風險套利。指數期貨的合理價 $F = S e^{(r_f - q)\tau}$ 來自「借錢買現股、領股利」的複製：期貨比現貨多了利息 r_f 、少了股利 q 。台股期每年夏天的逆價差，第一個原因就是除息旺季的 q 大於 r_f 。但套利的兩個方向難易不同：期貨太貴時，買現股、賣期貨很容易；期貨太便宜時，要放空整籃股票，成本高、限制多，所以逆價差可以偏離合理價很久。

KNOWLEDGE CHECK 6

TAIEX is at an illustrative 25,000, $r_f = 2\%$, the futures expires in 0.25 years, and dividends worth $q\tau = 0.3\%$ of the index go ex before expiry. Compute the fair futures price and the basis. Then explain what happens to the basis on the ex-dates, other things equal.

ANSWER

$F = 25,000 e^{0.005 - 0.003} = 25,050.05$, a basis of +50.05 points; without the dividends it would be +125.31. On an ex-date the index drops by the dividend, while the futures price, which already excluded it, does not move. The basis therefore rises by the dividend: discounts that reflect dividends shrink as the stocks go ex.

8

THE TX BASIS THROUGH THE 2025 DIVIDEND SEASON

- ☐ decompose the TX-TAIEX basis into carry, dividends and a residual, and calculate the dividend yield and financing rate it implies

Equation (6.24) leaves one degree of freedom: choose r_f to learn q , or q to learn r_f . Deciding which reading fits needs the dividends themselves, measured outside the futures market, and TWSE effectively publishes them every day.

Measuring the dividends

Besides the price index S , TWSE computes a total-return TAIEX, TR (報酬指數), that reinvests cash dividends (TWSE 2023). On an ex-dividend date the price index falls by the dividends of the stocks going ex; the total-return index does not. If dividends worth a fraction δ of the previous close go ex, the ratio TR/S therefore rises by the factor $1/(1 - \delta)$. Over the life of a contract,

$$q_{\text{real}}\tau = \ln \frac{\text{TR}_T}{\text{TR}_t} - \ln \frac{S_T}{S_t} = - \sum_{\text{ex-dates}} \ln(1 - \delta_j),$$

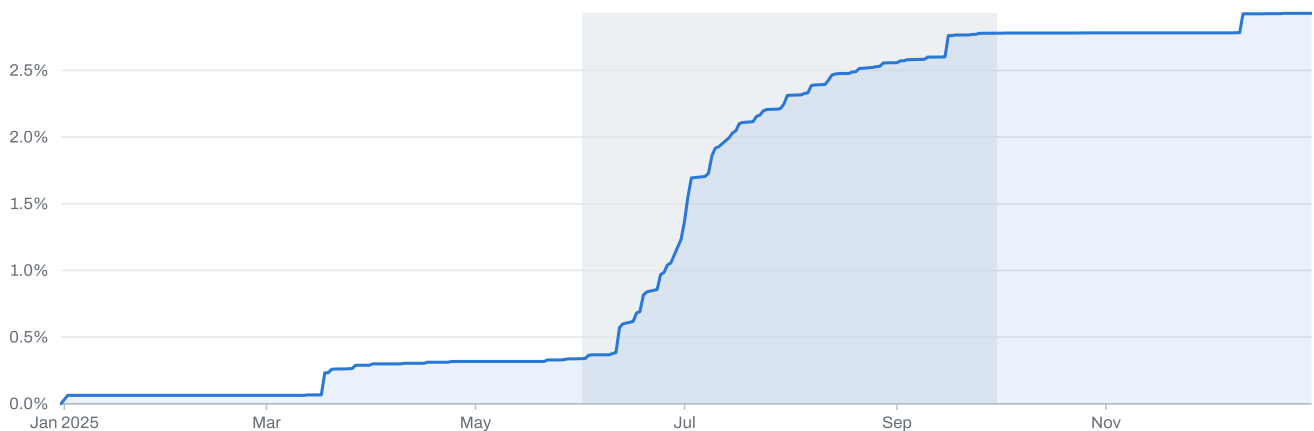
which is the $q\tau$ of (6.22) under proportional dividends, counted on their ex-dates.

TSMC's NT\$5 dividend went ex on 11 December 2025, inside the December contract's life, but was paid on 8 January 2026 (TSMC 2025). On the ex-date the ratio TR/S rose by 14.21 bp in log terms, so $\delta = 1 - e^{-0.00142149} = 14.205$ bp of the previous close, 28,400.73: 40.34 index points. The day before, the press had estimated the points removed by that day's ex-dividend stocks at the same 40.34, almost all from TSMC (money.udn.com).

Exhibit 8 accumulates the measure over 2025. Dividends going ex during the year came to $q\tau = 2.93\%$ of the index, 2.44% of it (83%) between June and September. July alone contributed 1.08%.

Exhibit 8: Cumulative TAIEX dividends by ex-date in 2025, as a share of the index

Cumulative dividends, $\ln(\text{TR}/\text{TR}_0) - \ln(S/S_0)$



Note: $\ln(\text{TR}_t/\text{TR}_0) - \ln(S_t/S_0)$ from 31 December 2024; shaded: June–September. The isolated steps of 18 March, 12 June, 16 September and 11 December fit TSMC's quarterly ex-dividend dates; the December one is confirmed above. Source: FinMind `TaiwanStockTotalReturnIndex` and `TaiwanStockPrice` (TAIEX), retrieved 10 October 2026.

Carry, dividends and a residual

With the dividends measured, (6.22) becomes an accounting identity. Let $F_{\text{fair}} = Se^{(r_f - q_{\text{real}})\tau}$ be the fair value given the dividends that actually went ex, and $\varepsilon = F - F_{\text{fair}}$ the residual. Then

$$F - S = \underbrace{S(e^{r_f\tau} - 1)}_{\text{carry}} - \underbrace{S e^{r_f\tau} (1 - e^{-q_{\text{real}}\tau})}_{\text{dividends}} + \underbrace{\varepsilon}_{\text{residual}}. \quad (6.25)$$

Carry is the interest on the index value at the 2% proxy rate. The dividend term applies the realized dividend yield to the index on the pricing date and carries it to expiry. The residual is everything a frictionless, perfect-foresight model leaves unexplained; a negative residual means the futures were cheap.

Exhibit 9: Decomposing the TX basis on 27 June 2025 (index points, $r_f = 2\%$)

Contract	Basis	Carry	Dividends	Residual	Implied $q\tau$	Realized $q\tau$	Implied financing rate
July	-458.08	23.52	223.34	-258.26	2.15%	0.99299%	-20.30%
August	-632.08	66.91	322.55	-376.44	3.14%	1.43449%	-9.49%
September	-746.08	101.68	383.79	-463.97	3.81%	1.70655%	-7.36%
December	-880.08	215.06	422.24	-672.90	4.92%	1.86971%	-4.44%

Note: basis = carry – dividends + residual, eq. (6.25); the implied financing rate is the r_f that makes $F = F_{\text{fair}}$. Implied $q\tau$ from (6.24) at $r_f = 2\%$; realized $q\tau$ from the total-return index between 27 June and expiry. Volumes: Exhibit 1.

Source: as Exhibit 1 and Exhibit 8.

Dividends explain 223.3 of the July contract's 458.1-point discount (Exhibit 9); carry pushes 23.5 points the other way, leaving a residual of -258.3. For December, carry adds 215 points and dividends explain 422 of 880, leaving a residual of -673: the residual grows with maturity. Read as financing rates, with dividends fixed at their realized values, the contracts priced money at -20.3% to -4.4% a year. Nobody lends at such rates.

The dividend column values the realized yields at the 27 June index level. TAIEX rose 22% by 17 December, so the points actually removed on the December contract's ex-dates were larger: 440.7, or 444.0 carried to expiry. Measured in points, the December residual would be -651 instead of -673.

Is the residual an artifact of one day? Partly. 27 June was the sample's extreme session: the December contract's discount, residual and implied financing rate were the most negative of all 146. TAIEX also closed at its high of the day.

By 30 June the December residual was -370 points. Over the nine sessions from 23 June to 3 July it averaged -461, and it was negative on 131 of the 146 sessions.

Back months add a second kind of noise: they trade thinly. On 27 June the July contract traded 60,785 lots but December only 49 (Exhibit 1). Over the sample, December traded a median of 14.5 lots a day.

Without a trade in the final minute, TAIFEX sets the settlement by rule. It can sit far from the last trade: 256 points for December on 11 July, when one lot traded. On 30 June the June 2026 contract did not trade

at all, yet settled at 21,501. A residual that grows with maturity on one day is therefore weak evidence.

A cleaner test avoids the index altogether: calendar spreads compare two settlement prices. On 27 June they implied forward dividends of 0.98% from the July to the August expiry, 0.67% to September and 1.11% to December. Just 0.44%, 0.27% and 0.16% went ex. Only the first spread rests on two actively traded contracts.

The season, month by month

Exhibit 10: Near-month TX contract by calendar month, 2025 (averages, index points)

Month	Trading days	Days to expiry	Basis	Carry	Dividends	Residual	Implied $q\tau$	Realized $q\tau$
April	20	18.8	-65	20	1	-84	0.445%	0.003%
May	20	14.7	-78	17	27	-68	0.448%	0.126%
June	21	14.8	-192	18	135	-75	0.953%	0.611%
July	23	18.3	-79	23	68	-34	0.445%	0.297%
August	21	14.9	-33	20	34	-19	0.221%	0.142%
September	21	14.7	0	20	24	+4	0.083%	0.097%
October	20	19.2	+67	29	0	+38	-0.136%	0.000%

Note: near month = nearest contract with a settlement price (on an expiry day, the next month). It traded a median of about 62,000 lots a day. Basis = carry – dividends + residual. Over 138 same-contract pairs of consecutive days, daily changes in the near-month basis have a sample standard deviation of 83.4 points (divisor 137). Source: as Exhibit 1 and Exhibit 8.

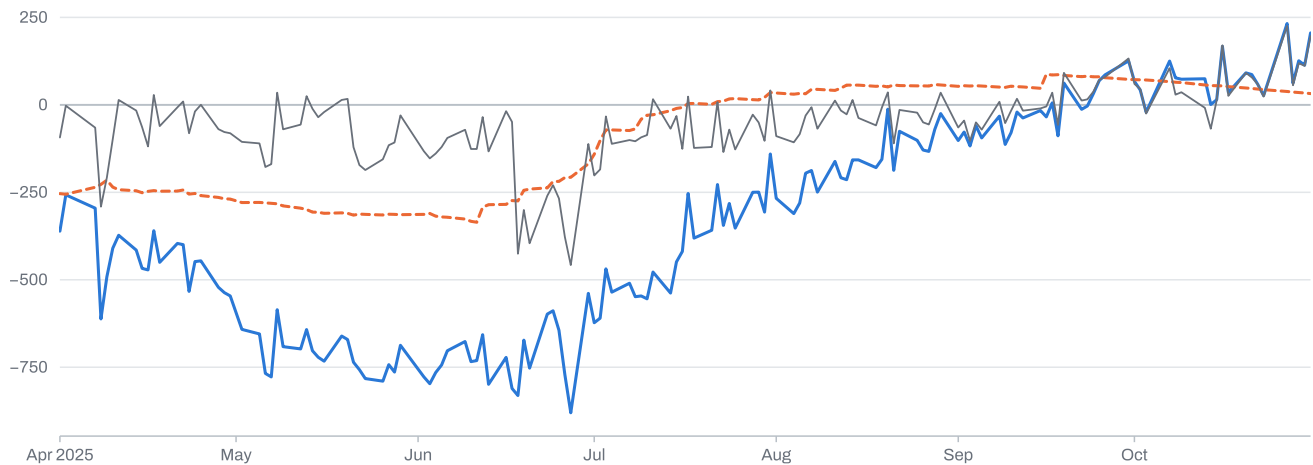
Dividends explain the shape of the season (Exhibit 10). In June they account for 135 points of the 192-point average discount, with carry offsetting 18 and the residual adding 75. In April, when almost nothing went ex, the residual of -84 exceeded the whole 65-point discount, because carry pushed the other way. The residual stayed between -84 and -68 until June, then narrowed, turning positive in September (+4) and reaching +38 in October.

The December contract, whose life spans the whole season, shows the pattern most clearly (Exhibit 11). Its residual averaged -191.5 points in April, -450.4 in June, -213.4 in August and +30.9 in October. As an implied financing rate it moved from -2.15% in June to +2.82% in October.

Exhibit 11: December 2025 TX contract: actual basis and fair basis with the dividends that went ex

— December contract: actual basis - - - December contract: fair basis (2% financing rate, dividends that went ex)
 — Near-month contract: actual basis

Basis $F - S$ (index points)



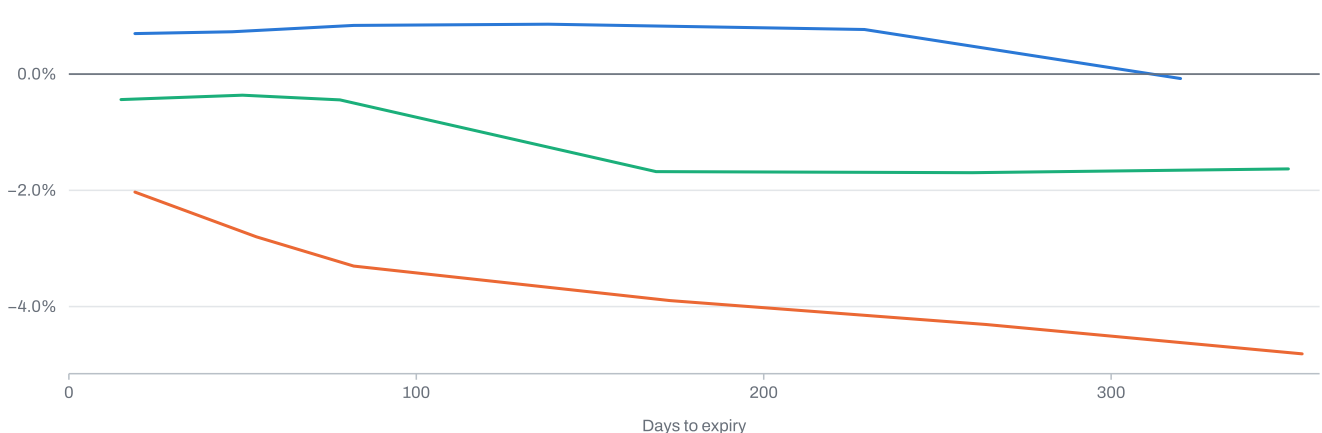
Note: fair basis $F_{\text{fair}} - S$ from (6.25) with $r_f = 2\%$; the gap between the two December lines is the residual. The December settlements rest on a median of 14.5 lots a day. Thin grey line: the near-month contract, which rolls to the next month at each expiry (the third Wednesday). Source: as Exhibit 1 and Exhibit 8.

Exhibit 12 shows the whole curve on three days. On 1 April contracts expiring in September or later sat about 270 points below the spring contracts, pricing the coming dividends. On 27 June every contract was deep below the index, and the curve kept falling with maturity well beyond the dividend season. By 31 October it was above the index for every contract except September 2026, whose life includes the 2026 season.

Exhibit 12: The TX curve before, during and after the dividend season (basis as % of TAIEX)

— 1 April 2025 — 27 June 2025 — 31 October 2025

Basis $(F - S)/S$



Note: each point is one contract's settlement minus the TAIEX close, divided by the close, against days to expiry. Beyond the second month, median daily volume was below 40 lots (Exhibit 1). TAIEX closed at 21,280.17 on 1 April, 22,580.08 on 27 June and 28,233.35 on 31 October. Source: as Exhibit 1.

Why the residual can persist

The no-arbitrage band of Section 7 is asymmetric. A rich TX is arbitrated with a stock purchase and a futures sale. A cheap one needs a short sale of every stock in the index, at an effective rate r_ℓ possibly far below 2%. Yet even at $r_f = 0$ the December contract sat below fair value on 43 of the 44 sessions in June and July (Example 9).

The 2025 pattern is consistent with hedging pressure against that weak lower bound, a hypothesis rather than a measured cause. The near-month residual was deepest just after the April tariff shock (−298 points on 8 April). Holders of Taiwan stocks then had reason to sell futures rather than shares. The longer contracts' residuals were deepest in May and June.

Near-month and December residuals both turned positive in a rising October market. MacKinlay and Ramaswamy (1988) found that S&P 500 futures mispricing grows, on average, with time to maturity.

For a basis trader the residual is the opportunity, but not a riskless one. Dividends must be forecast and the stock leg must track TAIEX. The residual can widen before expiry forces convergence (LM25 models it as a mean-reverting spread).

EXAMPLE 9

The December 2025 contract on 27 June 2025

On 27 June 2025 TAIEX closed at 22,580.08 and the December contract, 173 days from expiry, settled at 21,700. Between 27 June and 17 December the total-return index implies realized dividends of $q_{\text{real}}\tau = 1.86971\%$.

(a) Compute the implied dividend yield at $r_f = 2\%$: as $q\tau$, as an annual rate and as the present value of dividends in index points. (b) Compute the fair value with the dividends that actually went ex, the residual and the implied financing rate. (c) How sensitive is (a) to the choice of r_f ?

SOLUTION

(a) $\tau = 173/365 = 0.47397$. By (6.24), $q\tau = 0.02 \times 0.47397 - \ln(21,700/22,580.08) = 0.00948 + 0.03976 = 4.92\%$, or 10.39% a year. In the discrete form (6.23), the present value of the implied dividends is $S - Fe^{-r_f\tau} = 1,084.8$ points, 4.80% of the index.

(b) $F_{\text{fair}} = 22,580.08 e^{0.0094795 - 0.0186971} = 22,372.90$, so the residual is $21,700 - 22,372.90 = -672.90$ points, 2.98% of the index. The financing rate that makes $F = F_{\text{fair}}$ is $(0.0186971 + \ln(21,700/22,580.08))/0.47397 = -4.44\%$.

(c) The implied $q\tau = r_f\tau - \ln(F/S)$ rises by τ per unit of r_f , so each percentage point of r_f moves it by 0.47 percentage points:

r_f	0%	1%	2%	3%
Implied $q\tau$	3.98%	4.45%	4.92%	5.40%
Residual (points)	-461.8	-567.1	-672.9	-779.2

No plausible financing rate rescues the model: even at $r_f = 0$ the residual is -461.8 points.

PITFALL Annualizing a noisy basis

TAIEX closes on a 13:25–13:30 call auction. The TX daily settlement is the volume-weighted average price of the last minute before the 13:45 close (TAIFEX). The daily basis therefore mixes two prices fifteen minutes apart.

On 30 June TAIEX fell 324 points (1.44%) to its low of the day, while the July contract settled 22 points higher. Its basis jumped from -458 to -112. About 40 of those points were ex-dividend: the total-return index fell only 1.26%, and an ex-date raises the basis by the dividend. The other 306 points were mostly timing noise: TAIEX rebounded 298 points on 1 July.

Annualizing magnifies such noise as expiry nears. On 15 April, one day before expiry, a -64-point basis implies a dividend yield of 119% a year. Average over several days, use calendar spreads, or measure $q\tau$ over the contract's life without annualizing.

導讀 把逆價差拆成三塊：持有成本、股利、殘差

用報酬指數減掉價格指數，可以量出每檔合約存續期間實際除息的股利，再把基差拆成「持有成本（利息）－股利＋殘差」。2025 年 6 月近月合約平均逆價差 192 點，其中 135 點來自除息，說明夏天的逆價差主要來自除息旺季。6/27 是整段樣本最極端的一天：12 月合約逆價差 880 點，持有成本 +215 點、除息 -422 點，剩下的殘差 -673 點；以 6/27 為中心的 9 個交易日平均殘差約 -461 點，而且就算把利率設成 0，6、7 月 44 個交易日中仍有 43 天期貨低於合理價。近月殘差在四月關稅衝擊後最深，較遠月份則在五、六月最深；到了十月殘差轉為正值，期貨反而高於合理價。遠月合約一天只成交幾口到幾十口，結算價常由交易所規則決定，解讀遠月殘差要打折扣。

KNOWLEDGE CHECK 7

On 27 June 2025 the September contract (82 days to expiry) settled at 21,834 and the December contract (173 days) at 21,700. With $r_f = 2\%$, compute the implied forward dividend yield $q_{12}(\tau_2 - \tau_1)$ between the two expiries. Why is this number free of the 13:30/13:45 timing problem, and why should it still be read with care?

ANSWER

$\tau_2 - \tau_1 = 91/365 = 0.24932$, so $q_{12}(\tau_2 - \tau_1) = 0.02 \times 0.24932 - \ln(21,700/21,834) = 0.004986 + 0.006156 = 1.11\%$, against 0.16% that went ex between 17 September and 17 December. Both inputs are TX settlement prices, so the TAIEX close never enters. But the two contracts traded only 22 and 49 lots that day, so their settlements are partly set by exchange rules.

9 TIME VALUE OF MONEY IN PYTHON

☐ implement time-value-of-money calculations in Python

Three habits keep TVM code honest. Write one function per formula, with its equation number in the docstring. Check the bracket before any numerical search. Use Polars expressions, not Python loops, for market data.

PYTHON

```
def annuity_pv(pmt, i, n, due=False):
    """PV of n level payments PMT at periodic rate i, eq. (6.5); due: (6.7)."""
    pv = pmt * n if i == 0 else pmt * (1 - (1 + i) ** -n) / i
    return pv * (1 + i) if due else pv

def bond_price(y, c, n, M=100.0):
    """Annual-coupon bond with coupon rate c and face value M, eq. (6.14)."""
    return annuity_pv(c * M, y, n) + M * (1 + y) ** -n

def bisect(fn, lo, hi, tol=1e-8):
    """Root of fn on a sign-changing [lo, hi]; each pass halves the bracket."""
    f_lo = fn(lo)
    if f_lo * fn(hi) > 0:
        raise ValueError("root not bracketed")
    while (hi - lo) / 2 > tol:
        mid = (lo + hi) / 2
        f_mid = fn(mid)
        if f_lo * f_mid <= 0:
            hi = mid
        else:
            lo, f_lo = mid, f_mid
    return (lo + hi) / 2

ytm = bisect(lambda y: bond_price(y, 0.015, 10) - 97.20, 0.0, 0.05)
print(f"{ytm:.6%}") # 1.808601%
print(f"{annuity_pv(1_000, 0.03, 10, due=True):.2f}") # 8,786.11
```

IN PYTHON Companion lab

The notebook `LM06_lab.ipynb` rebuilds every exhibit and computational example, including the amortization schedules and the volume and robustness checks of Section 8. It solves the computational practice problems and ends with an extension exercise on another dividend season.

The notebook reads FinMind files that you download once with your own token. From the repository root, run `uv run python data/fetch_finmind.py`, then `uv run python data/make_lab_data.py`. FinMind's licence does not allow the book to redistribute the data. After that, the notebook runs offline.

SUMMARY

- An arithmetic sequence adds a constant; a geometric sequence multiplies by a constant ratio. The geometric sum (6.2) tends to $a_1/(1 - x)$ only when $|x| < 1$, and the share of the infinite sum beyond

term n is exactly x^n .

- Discounting makes every level stream a geometric series with ratio $v = 1/(1+i)$, giving the annuity factors (6.5)–(6.6). An annuity due is worth $1+i$ times as much, (6.7).
- Growing payments give (6.9) and (6.10), which require $\gamma < i$. The Gordon model $P_0 = D_1/(k - \gamma)$ reads a price as an implied return. For 0050 on 8 October 2026 it gives $k = 9.94\%$ at 8.43% dividend growth, with two thirds of the value beyond year 30.
- A bond trades above par exactly when its coupon rate exceeds its yield, (6.15). The yield to maturity is unique for conventional cash flows. Bisection finds it by halving the bracket, and so the error bound, at each step.
- A loan balance is the present value of the remaining payments, (6.17), and principal portions grow at the rate i , (6.18). On the illustrative NT\$10 million mortgage, equal-principal repayment pays NT\$360,064 less interest (undiscounted) and a 3-year grace period NT\$261,984 more. Yet at the loan rate every schedule is worth exactly the principal.
- No-arbitrage gives forward rates (6.20), covered interest parity (6.21) and the futures fair value $F = Se^{(r_f - q)\tau}$, (6.22); observed prices imply $r_f - q$, (6.24).
- Dividends going ex in 2025 came to $q\tau = 2.93\%$ of TAIEX, 83% of it from June to September: the summer shape of the TX discount. On 27 June, the extreme session, the December contract's 880-point discount split into 215 points of carry, 422 of dividends and a -673 residual. The residual averaged -461 over 23 June–3 July. Even at a zero financing rate the contract sat below fair value on 43 of 44 June–July sessions.

PRACTICE PROBLEMS

1. The sum $1 + 0.9 + 0.9^2 + \dots + 0.9^{19}$ is *closest* to:
A. 8.78
B. 10.00
C. 20.00
2. A saver deposits NT\$5,000 at the end of each month for 10 years in an account paying 0.15% a month. Compute the balance after the last deposit and the part of it that is interest. What is the deposits' present value?
3. NT\$2,000 is received at the start of each year for 15 years and money earns 2.5% a year. The present value is *closest* to:
A. NT\$24,763
B. NT\$25,382
C. NT\$30,000
4. A stock trades at NT\$80 and has just paid a dividend of NT\$3.20. With the Gordon model, compute the growth implied by an 8% required return and the required return implied by 4% growth.

5. A company pays out 60% of its earnings. Its dividends are expected to grow at 5% a year forever and the required return is 9%. Its justified forward P/E is *closest* to:
 - A. 12.00
 - B. 15.00
 - C. 15.75
6. A 7-year bond pays a 2.25% annual coupon on face 100 and yields 1.75%. Its price is *closest* to:
 - A. 96.73
 - B. 100.00
 - C. 103.27
7. A 3-year bond with a 2% annual coupon trades at 98.00. Starting from the bracket $[2\%, 4\%]$, perform three bisection steps for its yield to maturity and report the bracket after each step. What is the yield, in percent to four decimal places?
8. An NT\$8,000,000 mortgage runs 20 years at 2.1% a year, paid monthly. Compute the payment, the balance after 8 years, and the split of payment 97 into interest and principal.
9. For the loan in Problem 8, compute the first and last payments and the total interest under equal-principal repayment. Compare the total interest with the level-payment loan's.
10. USD/TWD trades at 31.20, and three-month money-market rates are 1.5% a year in TWD and 4.2% a year in USD (simple interest; illustrative). Compute the no-arbitrage three-month forward rate and the forward points. Which currency trades at a forward premium?
11. TAIEX is at an illustrative 23,000 and a futures contract expires in 0.2 years. With $r_f = 2\%$, dividends worth $q\tau = 1.2\%$ of the index will go ex before expiry. Compute the fair futures price and basis. If the futures trades at 22,600, compute the implied $q\tau$, the residual and the implied financing rate.
12. (*Python*) Using the lab data, decompose the basis of the September 2025 contract (202509) into carry, dividends and residual. Average each by calendar month, April to September 2025 (September: 12 sessions to 16 September; drop the expiry-day row, whose settlement price is 0). Is the shape of its residual over the months, when it is deepest, closer to the near-month contract's or to the December contract's? Why must the residual shrink as expiry approaches?

SOLUTIONS

1. A is correct. By (6.2), $(1 - 0.9^{20})/(1 - 0.9) = 8.7842$. B is the infinite sum $1/(1 - 0.9)$; C treats the 20 terms as if they did not shrink.

2. By (6.6), $FV = 5,000 \times (1.0015^{120} - 1)/0.0015 = \text{NT\$}656,853.04$. Deposits total $\text{NT\$}600,000$, so interest is $\text{NT\$}56,853.04$. By (6.5) the present value is $\text{NT\$}548,723.77$.
3. **B** is correct. By (6.5) and (6.7), $2,000 \times a_{15,2.5\%} \times 1.025 = 25,381.82$. A is the ordinary annuity, which places every payment a year too late; C ignores discounting.
4. By (6.13), $\gamma = (0.08 \times 80 - 3.20)/(80 + 3.20) = 3.20/83.20 = 3.846\%$; check: $3.20 \times 1.03846/(0.08 - 0.03846) = 80.00$. By (6.12), $k = 3.20 \times 1.04/80 + 0.04 = 4.16\% + 4\% = 8.16\%$.
5. **B** is correct. $P_0/E_1 = 0.60/(0.09 - 0.05) = 15.00$. C is the trailing P/E, $0.60 \times 1.05/0.04 = 15.75$; A divides the payout by γ instead of $k - \gamma$.
6. **C** is correct. By (6.14), $P = 2.25 \times a_{7,1.75\%} + 100 \times 1.0175^{-7} = 103.2673$. By (6.15), $P - 100 = (2.25 - 1.75) \times a_{7,1.75\%} = +3.27$, so A has the sign of the coupon surplus reversed; B would require the coupon rate to equal the yield.
7. Step 1: midpoint 3.00%, $P - 98.00 = -0.8286 < 0$, so the yield is below 3%: bracket [2.00%, 3.00%]. Step 2: 2.50%, +0.5720: bracket [2.50%, 3.00%]. Step 3: 2.75%, -0.1317: bracket [2.50%, 2.75%]. Continuing, the yield is 2.7030%.
8. With $i = 0.175\%$ and $n = 240$, (6.16) gives $\text{PMT} = \text{NT\$}40,850.63$ (40,850.632422 unrounded). By (6.17), $B_{96} = \text{PMT } a_{144,i} = \text{NT\$}5,195,828.36$. Payment 97 contains interest $iB_{96} = \text{NT\$}9,092.70$ and principal $\text{NT\$}31,757.93$, which equals $\text{PMT } v^{144}$ by (6.18). Total interest over the life is $240 \text{PMT} - B_0 = \text{NT\$}1,804,151.78$.
9. Principal is $8,000,000/240 = \text{NT\$}33,333.33$ a month, so the first payment is $33,333.33 + 14,000.00 = \text{NT\$}47,333.33$ and the last $(8,000,000/240) \times 1.00175 = \text{NT\$}33,391.67$. By (6.19), total interest is $0.00175 \times 8,000,000 \times 241/2 = \text{NT\$}1,687,000.00$, $\text{NT\$}117,151.78$ less than the level-payment loan's $\text{NT\$}1,804,151.78$. The gap is undiscounted: at the 2.1% loan rate both schedules are worth $\text{NT\$}8,000,000$. Discounted at 1.7% a year (0.017/12 a month), equal principal is $\text{NT\$}18,811$ cheaper, at 3% $\text{NT\$}37,294$ dearer.
10. By (6.21), $F = 31.20 \times (1 + 0.015 \times 0.25)/(1 + 0.042 \times 0.25) = 30.9916$, forward points -0.2084 TWD per USD. TWD, the lower-yielding currency, trades at a forward premium: a forward buys fewer TWD per USD than spot.
11. $F_{\text{fair}} = 23,000 e^{0.004 - 0.012} = 22,816.73$, a basis of -183.27 points. At $F = 22,600$, by (6.24), $q\tau = 0.004 - \ln(22,600/23,000) = 2.1544\%$; the residual is $22,600 - 22,816.73 = -216.73$ points; and the implied financing rate is $(0.012 + \ln(22,600/23,000))/0.2 = -2.77\%$.
12. Monthly averages in index points (basis = carry - dividends + residual):

Month	Basis	Carry	Dividends	Residual
April	-439	165	478	-126
May	-637	145	516	-266
June	-619	113	459	-273
July	-253	79	178	-154
August	-73	42	74	-41
September	-34	12	40	-6

The shape follows the December contract's, not the near month's (deepest in April). The residual is deepest in May and June, at about -270 points, and decays as expiry approaches. As $\tau \rightarrow 0$, carry and dividends vanish; F converges to the final settlement price, the average index in the last half hour of 17 September. The basis, and with it the residual, shrinks to the small gap between that average and the close.

GLOSSARY

Term	中文	Meaning
Ordinary annuity	普通年金（期末年金）	Level payments at the end of each period, (6.5).
Annuity due	期初年金	Level payments at the start of each period, (6.7).
Perpetuity; growing perpetuity	永續年金；成長型永續年金	Level payments forever, worth PMT/i ; payments growing at $\gamma < i$, worth $PMT_1/(i - \gamma)$.
Gordon growth model	高登成長模型	$P_0 = D_1/(k - \gamma)$, (6.11); read backwards, $k = D_1/P_0 + \gamma$.
Yield to maturity	到期殖利率	Single discount rate equating the promised cash flows to the price.
Internal rate of return	內部報酬率	Rate that sets the NPV to zero.
Level-payment / equal-principal loan	本息平均攤還／本金平均攤還	Equal payments, (6.16) / equal principal plus interest, (6.19).
Spot rate; implied forward rate	即期利率；隱含遠期利率	Zero-coupon yield; future-period rate implied by spot rates, (6.20).
Covered interest parity	拋補利率平價	No-arbitrage link between spot, forward and two interest rates, (6.21).
Cash-and-carry arbitrage	現貨持有套利	Buying the underlying and selling the futures (reverse: the opposite).
Basis; discount / premium	基差；逆價差／正價差	$F - S$; futures below / above the index.
Implied dividend yield	隱含股利率	q solving $F = Se^{(r_f - q)\tau}$ at a given r_f , (6.24).
Implied financing rate	隱含資金成本	r_f solving the same equation at a given q ; the implied repo rate.
Total-return index	報酬指數	Index that reinvests cash dividends on their ex-dates.

Term	中文	Meaning
Final settlement price	最後結算價	Cash-settlement price of an expiring TX contract: the average index over the last 30 minutes of the expiry day.

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